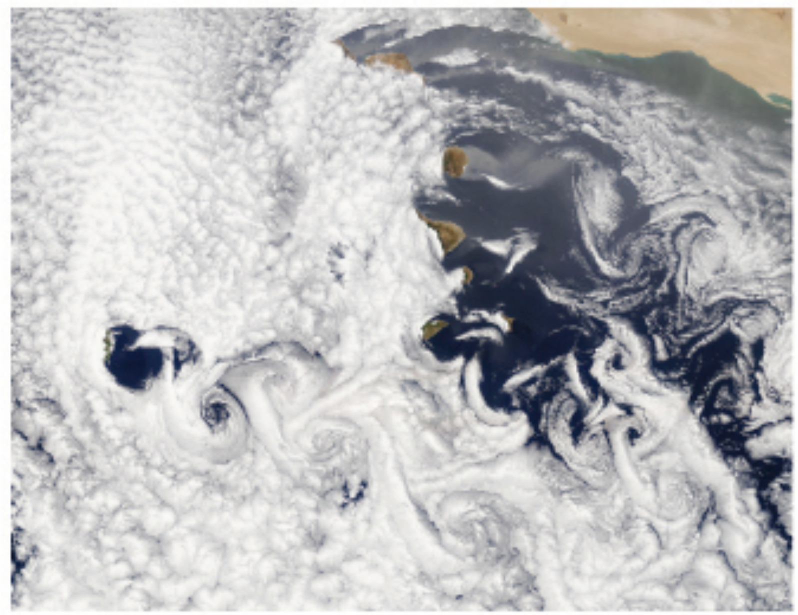
Ocean Modeling - EAS 8803

Geostrophy and Shallow-water model

- Further scaling of the primitive equations of motion
- Dimensionless number and active dynamics
- Steady and homogenous flows (***geostrophic flow***)
- Properties of geostrophic flows
- Unsteady and homogenous flows (***shallow water model***)

Introduction to
Geophysical Fluid Dynamics
Physical and Numerical Aspects



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Academic Press

Chapter 7

Primitive Equations describing motions of geophysical flows

$$
\begin{aligned}
x-\text{momentum:}\quad & \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} - f v = \\
& -\frac{1}{\rho_0}\,\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right) \\
y-\text{momentum:}\quad & \frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z} + f u = \\
& -\frac{1}{\rho_0}\,\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial v}{\partial z}\right) \\
z-\text{momentum:}\quad & 0 \;=\; -\,\frac{\partial p}{\partial z} \;-\; \rho g \\
\text{continuity:}\quad & \frac{\partial u}{\partial x} \;+\; \frac{\partial v}{\partial y} \;+\; \frac{\partial w}{\partial z} \;=\; 0 \\
\text{energy:}\quad & \frac{\partial \rho}{\partial t} + u\,\frac{\partial \rho}{\partial x} + v\,\frac{\partial \rho}{\partial y} + w\,\frac{\partial \rho}{\partial z} = \\
& \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial \rho}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial \rho}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa_E\frac{\partial \rho}{\partial z}\right)\;,
\end{aligned}
$$

Primitive Equations describing motions of geophysical flows

$$x - \text{momentum:} \qquad \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} - fv =$$

$$-\frac{1}{\rho_0}\,\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right)$$

Scaling of terms

$$\boxed{\frac{U}{T}\,,\quad \frac{U^2}{L}\,,\quad \frac{U^2}{L}\,,\quad \frac{WU}{H}\,,}\ \boxed{\Omega U},\quad \frac{P}{\rho_0 L}\,,\quad \boxed{\frac{\mathcal{A}U}{L^2}\,,\quad \frac{\nu_E U}{H^2}}$$

inertial terms *rotation* *frictional forces*

divide by ΩU

$$\boxed{\frac{1}{\Omega T}\,,\quad \frac{U}{\Omega L}\,,\quad \frac{U}{\Omega L}\,,\quad \frac{WL}{UH}\cdot\frac{U}{\Omega L}},\ \boxed{1\,,}\quad \frac{P}{\rho_0 \Omega L U}\,,\quad \boxed{\frac{\mathcal{A}}{\Omega L^2}\,,\quad \frac{\nu_E}{\Omega H^2}}$$

Important Dimensionless Numbers

measuring the sizes of the terms in the equations

1 ***Rossby Number***
advection/rotation

$$Ro = \frac{U}{\Omega L}$$

2 ***Ekman Number***
viscous force/rotation

$$Ek = \frac{\nu_E}{\Omega H^2}$$

Typical Scales

Scale	Oceanic value
L	10 km = 10^4 m
H	100 m = 10^2 m
T	$\geq$ 1 day $\simeq 9 \times 10^4$ s
U	0.1 m/s
W	
P	variable
$\Delta\rho$	
Ω	10^{-5} s^{-1}.

inertial terms *rotation* *frictional forces*

$$\boxed{\frac{1}{\Omega T}, \quad \frac{U}{\Omega L}, \quad \frac{U}{\Omega L}, \quad \frac{WL}{UH} \cdot \frac{U}{\Omega L}}, \quad \boxed{1,} \quad \frac{P}{\rho_0 \Omega L U}, \quad \boxed{\frac{\mathcal{A}}{\Omega L^2}, \quad \frac{\nu_E}{\Omega H^2}}$$

Important Dimensionless Numbers

measuring the sizes of the terms in the equations

3 **Reynolds Number**

inertial forces/frictional forces

high = turbulent flows

$$Re = \frac{UL}{\nu_E} = \frac{U}{\Omega L} \cdot \frac{\Omega H^2}{\nu_E} \cdot \frac{L^2}{H^2} = \frac{Ro}{Ek} \left(\frac{L}{H}\right)^2$$

$$\frac{\textit{inertial terms}}{\textit{frictional forces}}$$

inertial terms *rotation* *frictional forces*

$$\boxed{\frac{1}{\Omega T}, \quad \frac{U}{\Omega L}, \quad \frac{U}{\Omega L}, \quad \frac{WL}{UH} \cdot \frac{U}{\Omega L}}, \quad \boxed{1,} \quad \frac{P}{\rho_0 \Omega L U}, \quad \boxed{\frac{\mathcal{A}}{\Omega L^2}, \quad \frac{\nu_E}{\Omega H^2}}$$

Important Dimensionless Numbers

measuring the sizes of the terms in the equations

4 *Richardson Number*

Available Potential Energy (APE)

$$Ri = \frac{gH\Delta\rho}{\rho_0 U^2}$$

Kinetic Energy (KE)

inertial terms *rotation* *frictional forces*

$$\boxed{\frac{1}{\Omega T}, \quad \frac{U}{\Omega L}, \quad \frac{U}{\Omega L}, \quad \frac{WL}{UH} \cdot \frac{U}{\Omega L}}, \quad \boxed{1,} \quad \frac{P}{\rho_0 \Omega L U}, \quad \boxed{\frac{\mathcal{A}}{\Omega L^2}, \quad \frac{\nu_E}{\Omega H^2}}$$

Geostrophic flows or steady state solution

$$\boxed{Ro_T \ll 1, \quad Ro \ll 1, \quad Ek \ll 1}$$

$$x - \text{momentum:} \quad \underbrace{\cancel{\frac{\partial u}{\partial t}}}_{small} + u\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{small} + v\underbrace{\cancel{\frac{\partial u}{\partial y}}}_{small} + w\underbrace{\cancel{\frac{\partial u}{\partial z}}}_{small} - \boxed{fv =}$$

$$\boxed{-\frac{1}{\rho_0}\frac{\partial p}{\partial x}} + \underbrace{\cancel{\frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial u}{\partial x}\right)}}_{small} + \underbrace{\cancel{\frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial u}{\partial y}\right)}}_{small} + \underbrace{\cancel{\frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right)}}_{small}$$

inertial terms *rotation* *frictional forces*

$$\boxed{\frac{1}{\Omega T}, \quad \frac{U}{\Omega L}, \quad \frac{U}{\Omega L}, \quad \frac{WL}{UH} \cdot \frac{U}{\Omega L}}, \quad \boxed{1,} \quad \frac{P}{\rho_0 \Omega L U}, \quad \boxed{\frac{\mathcal{A}}{\Omega L^2}, \quad \frac{\nu_E}{\Omega H^2}}$$

Geostrophic flows or steady state solution

$$\boxed{Ro_T \ll 1, \quad Ro \ll 1, \quad Ek \ll 1}$$

$$x-\text{momentum:} \quad \underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{small}} + u\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\text{small}} + v\underbrace{\cancel{\frac{\partial u}{\partial y}}}_{\text{small}} + w\underbrace{\cancel{\frac{\partial u}{\partial z}}}_{\text{small}} - \boxed{fv =}$$

$$\boxed{-\frac{1}{\rho_0}\frac{\partial p}{\partial x}} + \underbrace{\cancel{\frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial u}{\partial x}\right)}}_{\text{small}} + \underbrace{\cancel{\frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial u}{\partial y}\right)}}_{\text{small}} + \underbrace{\cancel{\frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right)}}_{\text{small}}$$

$$z-\text{momentum:} \quad 0 = -\frac{\partial p}{\partial z} - \rho g$$

$$\text{continuity:} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\text{energy:} \quad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} =$$

$$\frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial \rho}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial \rho}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa_E\frac{\partial \rho}{\partial z}\right)$$

Geostrophic flows or steady state solution

$$\boxed{\begin{array}{r} Ro_T \ll 1, \quad Ro \ll 1, \quad Ek \ll 1 \\ \rho = 0 \text{ (no density variation)} \end{array}}$$

$$x-\text{momentum:} \quad \underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\textit{small}} + u \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\textit{small}} + v \underbrace{\cancel{\frac{\partial u}{\partial y}}}_{\textit{small}} + w \underbrace{\cancel{\frac{\partial u}{\partial z}}}_{\textit{small}} - \boxed{fv =}$$

$$\boxed{-\frac{1}{\rho_0}\frac{\partial p}{\partial x}} + \underbrace{\cancel{\frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial u}{\partial x}\right)}}_{\textit{small}} + \underbrace{\cancel{\frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial u}{\partial y}\right)}}_{\textit{small}} + \underbrace{\cancel{\frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right)}}_{\textit{small}}$$

$$z-\text{momentum:} \quad \underbrace{\cancel{0 = -\frac{\partial p}{\partial z} - \rho g}}_{\textit{vanish}}$$

$$\text{continuity:} \quad \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

$$\text{energy:} \quad \underbrace{\cancel{\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial \rho}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial \rho}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa_E\frac{\partial \rho}{\partial z}\right)}}_{\textit{vanish}}$$

Geostrophic flows or steady state solution

$$\boxed{\begin{array}{r} Ro_T \ll 1, \quad Ro \ll 1, \quad Ek \ll 1 \\ \rho = 0 \text{ (no density variation)} \end{array}}$$

$$\begin{aligned} -fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\ +fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\ 0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0, \end{aligned}$$

homogeneous flow

Properties of Geostrophic homogeneous flows

geostrophic flows follow lines of constant pressure = ***isobars***

$$u = \frac{-1}{\rho_0 f}\frac{\partial p}{\partial y}, \qquad v = \frac{+1}{\rho_0 f}\frac{\partial p}{\partial x}$$

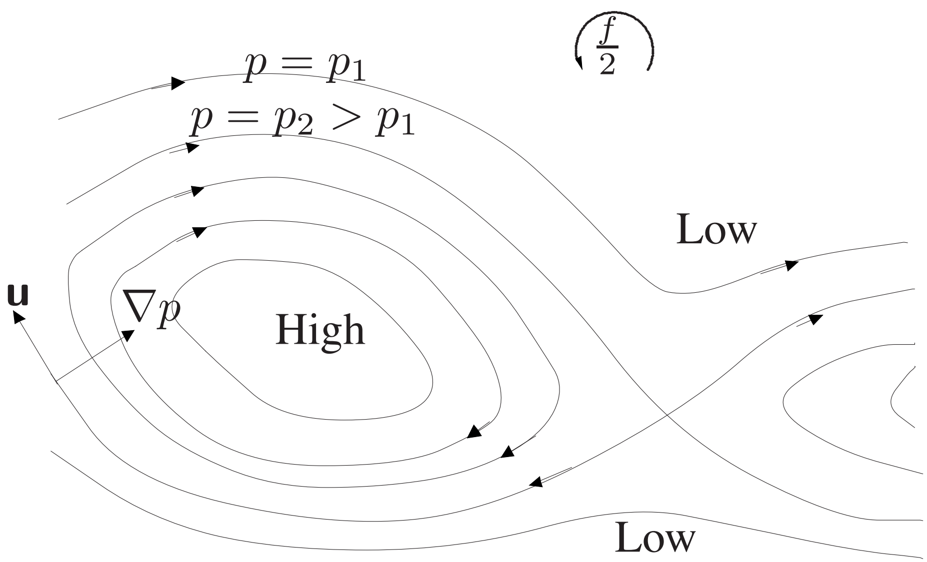


Figure 7-1 Example of geostrophic flow. The velocity vector is everywhere parallel to the lines of equal pressure. Thus, pressure contours act as streamlines. In the Northern Hemisphere (as pictured here), the fluid circulates with the high pressure on its right. The opposite holds for the Southern Hemisphere.

Properties of Geostrophic homogeneous flows

$$\frac{\partial}{\partial z}\left(\begin{aligned} -fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\ +fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\ 0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \end{aligned}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

no vertical shear

$$-f\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right) = -\frac{1}{\rho_0}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) = 0$$

Taylor–Proudman theorem (Taylor, 1923; Proudman, 1953)

Properties of Geostrophic homogeneous flows

if *f-plane*

$$\frac{\partial}{\partial x}\left(\begin{array}{rcl} -fv &=& -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x} \\ +fu &=& -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial y} \end{array}\right)$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

non-divergent

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{1}{\rho_0 f}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho_0 f}\frac{\partial p}{\partial x}\right) = 0$$

Properties of Geostrophic homogeneous flows

if f-plane

$$\frac{\partial}{\partial x}\left(\begin{array}{rcl} -fv &=& -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x} \\ +fu &=& -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial y} \end{array}\right)$$

$$\begin{aligned} 0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \end{aligned}$$

non-divergent

$$\frac{\partial w}{\partial z} = 0$$

Properties of Geostrophic homogeneous flows

if *f-plane*

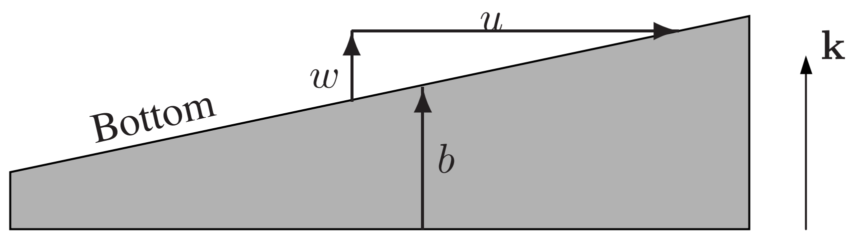
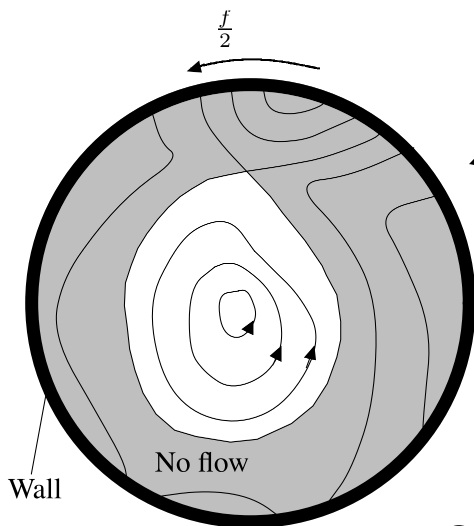


Figure 7-3 Schematic view of a flow over a sloping bottom. A vertical velocity must accompany flow across isobaths.



therefore flow is permitted only along isobaths

Figure 7-4 Geostrophic flow in a closed domain and over irregular topography. Solid lines are isobaths (contours of equal depth). Flow is permitted only along closed isobaths

$$\frac{\partial w}{\partial z} = 0$$

Properties of Geostrophic homogeneous flows

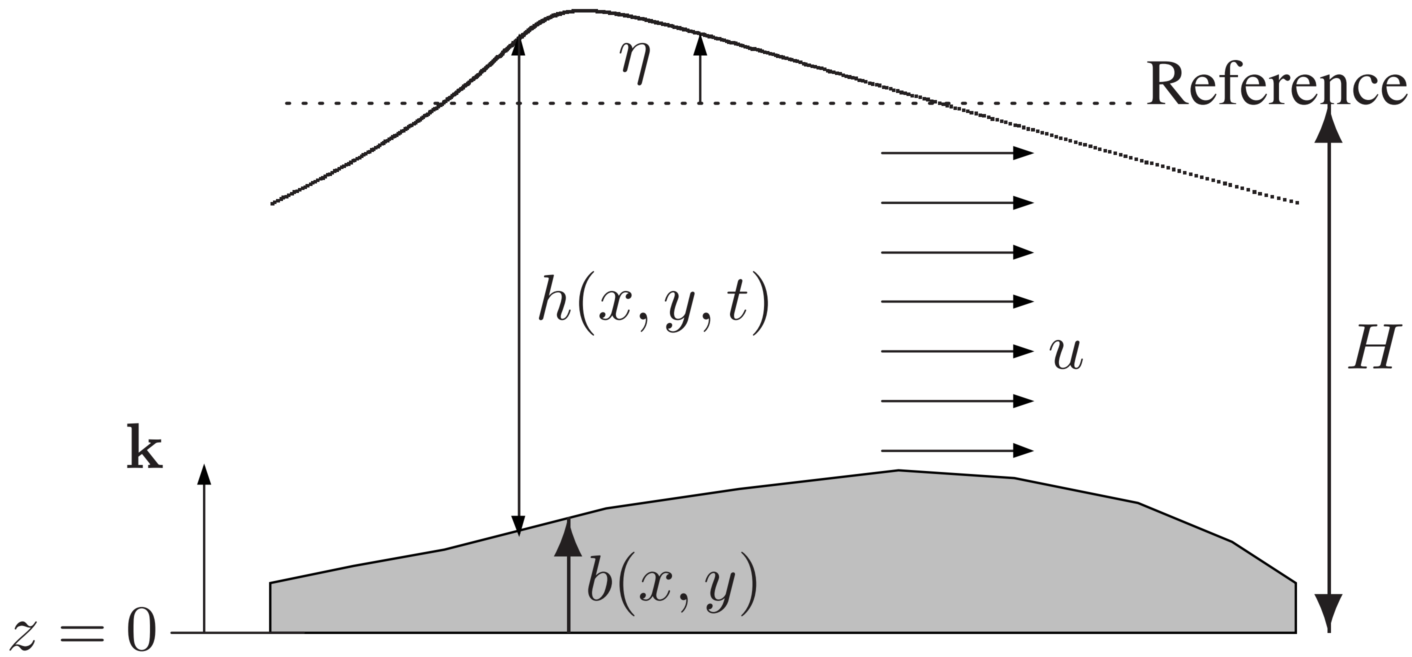
$$
\begin{aligned}
-fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
+fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0,
\end{aligned}
$$

homogeneous flow

if pressure only a function of the sea level

$$\boxed{p = \rho_0 g \eta}$$

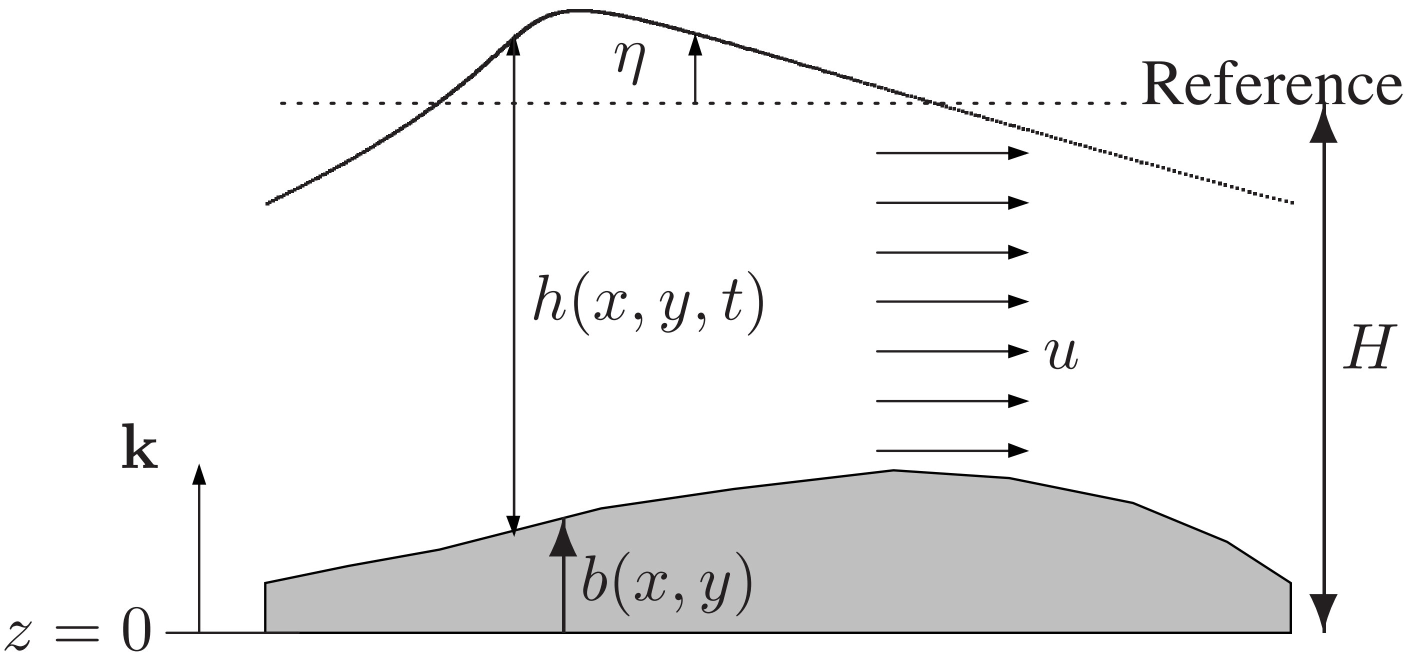
Properties of Geostrophic homogeneous flows



if pressure only a function of the sea level

$$\boxed{p = \rho_0 g \eta}$$

Properties of Geostrophic homogeneous flows



re-write the continuity equation for the layer velocity

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \right) \int_b^{b+h} dz$$

integrate from bottom to surface

Properties of Geostrophic homogeneous flows

$$
\boxed{
\begin{aligned}
w(z = b+h) &= \frac{\partial}{\partial t}(b+h) + u\frac{\partial}{\partial x}(b+h) + v\frac{\partial}{\partial y}(b+h) \\
&= \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \\
w(z=b) &= u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}\,.
\end{aligned}
}
$$

boundary conditions

where $\eta = b + h - H$

re-write the continuity equation for the layer velocity

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\int_b^{b+h} dz + \boxed{[w]_b^{b+h}}$$

integrate from bottom to surface

Properties of Geostrophic homogeneous flows

$$\boxed{\begin{aligned} w(z=b+h) &= \frac{\partial}{\partial t}(b+h) \;+\; u\frac{\partial}{\partial x}(b+h) \;+\; v\frac{\partial}{\partial y}(b+h) \\ &= \frac{\partial \eta}{\partial t} \;+\; u\frac{\partial \eta}{\partial x} \;+\; v\frac{\partial \eta}{\partial y} \\ w(z=b) &= u\frac{\partial b}{\partial x} \;+\; v\frac{\partial b}{\partial y}\,. \end{aligned}}$$

boundary conditions

where $\eta = b + h - H$

re-write the continuity equation for the layer velocity

$$\frac{\partial \eta}{\partial t} \;+\; \frac{\partial}{\partial x}(hu) \;+\; \frac{\partial}{\partial y}(hv) \;=\; 0.$$

NEW *continuity equation!*

Equations of Geostrophic homogeneous flows

if pressure only a function of the sea level

$$\boxed{p = \rho_0 g \eta}$$

$$-\, f v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$+\, f u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0,$$

Equations of Geostrophic homogeneous flows

$$- \; fv \;=\; -\,g\,\frac{\partial \eta}{\partial x}$$

$$+ \; fu \;=\; -\,g\,\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0,$$

Equations of Geostrophic homogeneous flows

shallow-water model or *barotropic equations*

describe unsteady motions of a 2D uniform density layer

or

of the depth average motion

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} - fv = -g\,\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + fu = -g\,\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$