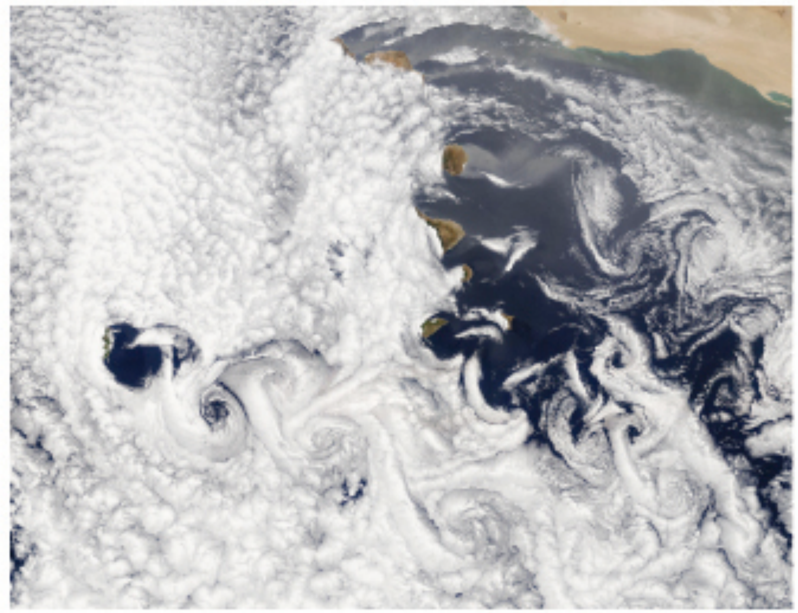
Ocean Modeling - EAS 8803

Equation of Motion

- Conservation equations (*mass, momentum, tracers*)
- Boussinesq Approximation
- Statistically average flow, Reynolds averaging
- Scaling the equations
- Primitive Equations for a geophysical flow

Introduction to
Geophysical Fluid Dynamics
Physical and Numerical Aspects



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Chapter 3-4

Mass Budget - *Continuity Equation*

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Boussinesq Approximation

$$\rho = \rho_0 + \rho'(x,\ y,\ z,\ t) \quad \text{with} \quad |\rho'| \ll \rho_0$$

Mass Budget - *Continuity Equation*

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Boussinesq Approximation

$$\rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \rho' \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \left(\frac{\partial \rho'}{\partial t} + u \frac{\partial \rho'}{\partial x} + v \frac{\partial \rho'}{\partial y} + w \frac{\partial \rho'}{\partial z} \right) = 0.$$

Continuity Equation in the ocean

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \qquad |\rho'| \ll \rho_0$$

Conservation of Volume

Equation of State (*Linear and Nonlinear*)

Linear

$$\rho = \rho_0 \left[1 - \alpha(T - T_0) + \beta(S - S_0)\right]$$

$$\rho_0 = 1028 \text{ kg/m}^3,$$

$$T_0 = 10°\text{C} = 283 \text{ K},$$

$$S_0 = 35,$$

Coefficients of ***thermal expansion*** *and* ***saline contraction***

$$\alpha = 1.7 \times 10^{-4} \text{ K}^{-1}$$

$$\beta = 7.6 \times 10^{-4}$$

Nonlinear (empirical)

$$\rho = \rho(T, S, p)$$

Equations for Tracers (*Temperature, Salinity and others*)

$$\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} + w\,\frac{\partial T}{\partial z} = \kappa_T\,\nabla^2 T,$$

advection-diffusion equation, where T can be any tracer

Flux Formulation of Tracer Equation

advective flux

$$\frac{\partial T}{\partial t} + \boxed{\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) + \frac{\partial}{\partial z}(wT)}$$

diffusive flux

$$\boxed{-\frac{\partial}{\partial x}\left(\kappa_T\frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(\kappa_T\frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial z}\left(\kappa_T\frac{\partial T}{\partial z}\right)} = 0$$

Momentum Budget - *Navier-Stokes Equations*

$$x: \quad \rho\left(\frac{du}{dt} + f_* w - f v\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{xy}}{\partial y} + \frac{\partial \tau^{xz}}{\partial z}$$

$$y: \quad \rho\left(\frac{dv}{dt} + f u\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau^{xy}}{\partial x} + \frac{\partial \tau^{yy}}{\partial y} + \frac{\partial \tau^{yz}}{\partial z}$$

$$z: \quad \rho\left(\frac{dw}{dt} - f_* u\right) = -\frac{\partial p}{\partial z} - \rho g + \frac{\partial \tau^{xz}}{\partial x} + \frac{\partial \tau^{yz}}{\partial y} + \frac{\partial \tau^{zz}}{\partial z},$$

Pressure Force | **Frictional Stresses**

Definition *Material Derivative*

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Momentum Budget - *Navier-Stokes Equations*

$$x: \quad \rho\left(\frac{du}{dt} + f_* w - f v\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{xy}}{\partial y} + \frac{\partial \tau^{xz}}{\partial z}$$

$$y: \quad \rho\left(\frac{dv}{dt} + f u\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau^{xy}}{\partial x} + \frac{\partial \tau^{yy}}{\partial y} + \frac{\partial \tau^{yz}}{\partial z}$$

$$z: \quad \rho\left(\frac{dw}{dt} - f_* u\right) = -\frac{\partial p}{\partial z} - \rho g + \frac{\partial \tau^{xz}}{\partial x} + \frac{\partial \tau^{yz}}{\partial y} + \frac{\partial \tau^{zz}}{\partial z},$$

Pressure Force **Frictional Stresses**

Viscous stress proportional to velocity gradients

$$\tau^{xx} = \mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right), \quad \tau^{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \tau^{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau^{yy} = \mu\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right), \quad \tau^{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau^{zz} = \mu\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\right), \qquad (3.$$

Momentum Budget - *Navier-Stokes Equations*

$$\frac{du}{dt} + f_* w - f v = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \nu \nabla^2 u$$

$$\frac{dv}{dt} + f u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} + \nu \nabla^2 v$$

$$\frac{dw}{dt} - f_* u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0} + \nu \nabla^2 w,$$

kinematic viscosity $\nu = \mu/\rho_0$

Only dynamic pressure is important to motion

$$p = p_0(z) + p'(x, y, z, t) \quad \text{with} \quad p_0(z) = P_0 - \rho_0 g z$$

Momentum Budget - *Navier-Stokes Equations*

$$\frac{du}{dt} + f_* w - f v = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \nu \nabla^2 u$$

$$\frac{dv}{dt} + f u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} + \nu \nabla^2 v$$

$$\frac{dw}{dt} - f_* u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0} + \nu \nabla^2 w,$$

Statistically average flow - *Reynolds averaging*

$$\frac{\partial \langle u \rangle}{\partial t} + \frac{\partial \langle uu \rangle}{\partial x} + \frac{\partial \langle vu \rangle}{\partial y} + \frac{\partial \langle wu \rangle}{\partial z} + f_* \langle w \rangle - f \langle v \rangle = -\frac{1}{\rho_0}\frac{\partial \langle p \rangle}{\partial x} + \nu \nabla^2 \langle u \rangle$$

$\langle \; \rangle$ *denotes time/space average*

Momentum Budget - *Navier-Stokes Equations*

$$\frac{du}{dt} + f_* w - f v = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \nu \nabla^2 u$$

$$\frac{dv}{dt} + f u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} + \nu \nabla^2 v$$

$$\frac{dw}{dt} - f_* u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0} + \nu \nabla^2 w,$$

Statistically average flow - *Reynolds averaging*

$$\frac{\partial \langle u \rangle}{\partial t} + \frac{\partial \langle uu \rangle}{\partial x} + \frac{\partial \langle vu \rangle}{\partial y} + \frac{\partial \langle wu \rangle}{\partial z} + f_* \langle w \rangle - f \langle v \rangle = -\frac{1}{\rho_0}\frac{\partial \langle p \rangle}{\partial x} + \nu \nabla^2 \langle u \rangle$$

introduce: $$u = \langle u \rangle + u'$$

with: $\langle u' \rangle = 0$ *average* *perturbation*

Momentum Budget - *Navier-Stokes Equations*

$$\frac{du}{dt} + f_* w - f v = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \nu \nabla^2 u$$

$$\frac{dv}{dt} + f u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} + \nu \nabla^2 v$$

$$\frac{dw}{dt} - f_* u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0} + \nu \nabla^2 w,$$

Statistically average flow - *Reynolds averaging*

$$\frac{\partial \langle u \rangle}{\partial t} + \frac{\partial(\langle u \rangle \langle u \rangle)}{\partial x} + \frac{\partial(\langle u \rangle \langle v \rangle)}{\partial y} + \frac{\partial(\langle u \rangle \langle w \rangle)}{\partial z} + f_* \langle w \rangle - f \langle v \rangle =$$

$$-\frac{1}{\rho_0}\frac{\partial \langle p \rangle}{\partial x} + \nu \nabla^2 \langle u \rangle - \frac{\partial \langle u'u' \rangle}{\partial x} - \frac{\partial \langle u'v' \rangle}{\partial y} - \frac{\partial \langle u'w' \rangle}{\partial z}.$$

turbulent fluctuations of the mean flow

Momentum Budget - *Navier-Stokes Equations + Reynolds averaging*

$$\frac{\partial}{\partial x}\left(\nu\,\frac{\partial \langle u\rangle}{\partial x} - \boxed{\langle u'u'\rangle}\right)\;,\;\frac{\partial}{\partial y}\left(\nu\,\frac{\partial \langle u\rangle}{\partial y} - \boxed{\langle u'v'\rangle}\right)\;,\;\frac{\partial}{\partial z}\left(\nu\,\frac{\partial \langle u\rangle}{\partial z} - \boxed{\langle u'w'\rangle}\right)$$

Reynolds stresses

Statistically average flow - *Reynolds averaging*

$$\frac{\partial \langle u\rangle}{\partial t} + \frac{\partial(\langle u\rangle\langle u\rangle)}{\partial x} + \frac{\partial(\langle u\rangle\langle v\rangle)}{\partial y} + \frac{\partial(\langle u\rangle\langle w\rangle)}{\partial z} + f_*\langle w\rangle - f\langle v\rangle =$$

$$-\frac{1}{\rho_0}\,\frac{\partial\langle p\rangle}{\partial x} + \boxed{\nu\,\nabla^2\langle u\rangle - \frac{\partial\langle u'u'\rangle}{\partial x} - \frac{\partial\langle u'v'\rangle}{\partial y} - \frac{\partial\langle u'w'\rangle}{\partial z}}\;.$$

turbulent fluctuations of the mean flow add to viscous stresses

Momentum Budget - *Navier-Stokes Equations + Reynolds averaging*

$$\frac{\partial}{\partial x}\left(\nu\,\frac{\partial\langle u\rangle}{\partial x}-\boxed{\langle u'u'\rangle}\right)\;,\;\frac{\partial}{\partial y}\left(\nu\,\frac{\partial\langle u\rangle}{\partial y}-\boxed{\langle u'v'\rangle}\right)\;,\;\frac{\partial}{\partial z}\left(\nu\,\frac{\partial\langle u\rangle}{\partial z}-\boxed{\langle u'w'\rangle}\right)$$

Reynolds stresses

horizontal eddy viscosity $\mathcal{A}$ vertical eddy viscosity ν_E

Statistically average flow - *Reynolds averaging*

$$\frac{\partial\langle u\rangle}{\partial t}+\frac{\partial(\langle u\rangle\langle u\rangle)}{\partial x}+\frac{\partial(\langle u\rangle\langle v\rangle)}{\partial y}+\frac{\partial(\langle u\rangle\langle w\rangle)}{\partial z}+f_*\langle w\rangle-f\langle v\rangle=$$

$$-\frac{1}{\rho_0}\,\frac{\partial\langle p\rangle}{\partial x}+\boxed{\frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial\langle u\rangle}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial\langle u\rangle}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_E\frac{\partial\langle u\rangle}{\partial z}\right)}$$

turbulent fluctuations of the mean flow
add to viscous stresses = eddy stresses

Scaling the equations

Variable	Scale	Unit	Atmospheric value	Oceanic value
x, y	L	m	100 km = 10^5 m	10 km = 10^4 m
z	H	m	1 km = 10^3 m	100 m = 10^2 m
t	T	s	$\geq \frac{1}{2}$ day $\simeq 4 \times 10^4$ s	≥ 1 day $\simeq 9 \times 10^4$ s
u, v	U	m/s	10 m/s	0.1 m/s
w	W	m/s		
p	P	kg/(m·s^2)	variable	
ρ	$\Delta\rho$	kg/m^3		

$$T \gtrsim \frac{1}{\Omega}$$

$$\frac{U}{L} \lesssim \Omega$$

$$H \ll L$$

$$W \ll U$$

Scales of Geophysical Flows

An example for the vertical component of momentum

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - f_* u =$$

small, small, small, small, small

$$\frac{W}{T}, \quad \frac{UW}{L}, \quad \frac{UW}{L}, \quad \frac{W^2}{H}, \quad \Omega U$$

Scales of terms

$$\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \quad - \frac{g\rho}{\rho_0} + \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial w}{\partial z}\right)$$

small, small, small

$$\frac{P}{\rho_0 H} \qquad \frac{g\Delta\rho}{\rho_0} \qquad \frac{\mathcal{A}W}{L^2} \qquad \frac{\mathcal{A}W}{L^2} \qquad \frac{\nu_E W}{H^2}$$

Scales of terms

Hydrostatic Balance

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_0}$$

Primitive Equations describing motions of geophysical flows

$$x-\text{momentum:}\quad \frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}+w\,\frac{\partial u}{\partial z}-fv=$$

$$-\frac{1}{\rho_0}\,\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right)$$

$$y-\text{momentum:}\quad \frac{\partial v}{\partial t}+u\,\frac{\partial v}{\partial x}+v\,\frac{\partial v}{\partial y}+w\,\frac{\partial v}{\partial z}+fu=$$

$$-\frac{1}{\rho_0}\,\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_E\frac{\partial v}{\partial z}\right)$$

$$z-\text{momentum:}\quad 0\;=\;-\frac{\partial p}{\partial z}\;-\;\rho g$$

$$\text{continuity:}\quad \frac{\partial u}{\partial x}\;+\;\frac{\partial v}{\partial y}\;+\;\frac{\partial w}{\partial z}\;=\;0$$

$$\text{energy:}\quad \frac{\partial \rho}{\partial t}+u\,\frac{\partial \rho}{\partial x}+v\,\frac{\partial \rho}{\partial y}+w\,\frac{\partial \rho}{\partial z}=$$

$$\frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial \rho}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial \rho}{\partial y}\right)+\frac{\partial}{\partial z}\left(\kappa_E\frac{\partial \rho}{\partial z}\right)\;,$$