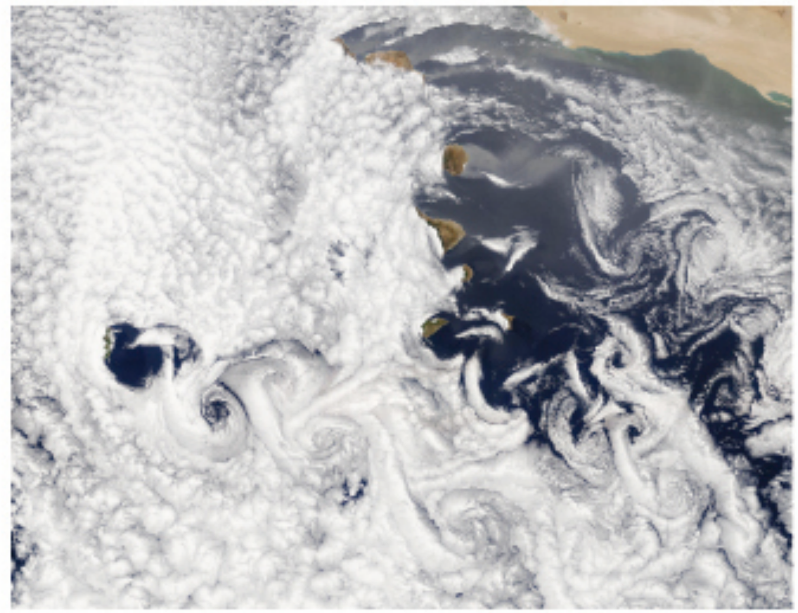
Ocean Modeling - EAS 8803

rotation

- We model the ocean on a rotating planet
- Rotation effects are considered through the Coriolis and Centrifugal Force
- The Coriolis Force arises because our reference frame (the Earth) is rotating
- **The Coriolis Force is the source of many interesting geophysical processes**

Introduction to
Geophysical Fluid Dynamics
Physical and Numerical Aspects



Benoit Cushman-Roisin and Jean-Marie Beckers

Academic Press

Chapter 2

A rotating framework - *The coordinates*

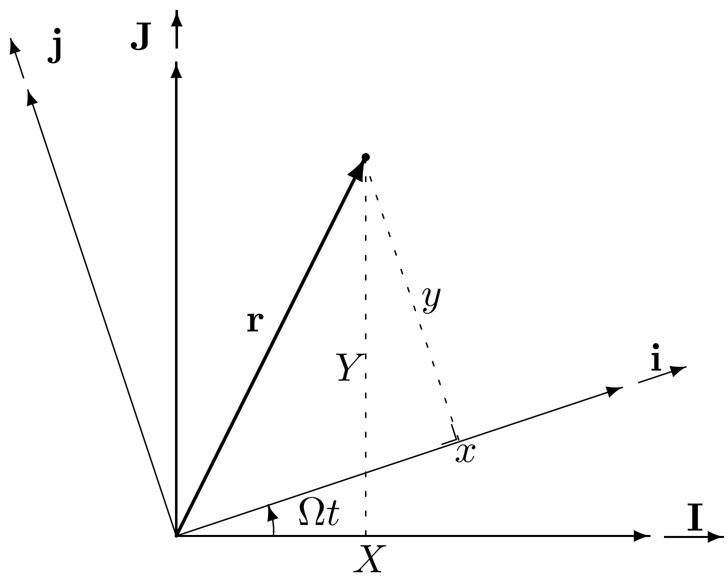


Figure 2-1 Fixed (X, Y) and rotating (x, y) frameworks of reference.

Rotating reference

Angular Velocity

$$x = +X \cos \Omega t + Y \sin \Omega t$$
$$y = -X \sin \Omega t + Y \cos \Omega t.$$

Fixed reference

A rotating framework - *The velocity (1st derivative)*

$$\frac{dx}{dt} = + \frac{dX}{dt}\cos\Omega t + \frac{dY}{dt}\sin\Omega t \overbrace{- \Omega X \sin\Omega t + \Omega Y \cos\Omega t}^{+\Omega y}$$

$$\frac{dy}{dt} = - \frac{dX}{dt}\sin\Omega t + \frac{dY}{dt}\cos\Omega t \underbrace{- \Omega X \cos\Omega t - \Omega Y \sin\Omega t}_{-\Omega x}$$

Relative Velocity

change of the
coordinate relative to
the moving frame

Absolute Velocity

$$\mathbf{u} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j}$$

$$\mathbf{U} = \frac{dX}{dt}\,\mathbf{I} + \frac{dY}{dt}\,\mathbf{J}$$

A rotating framework - *The velocity (1st derivative)*

$$\underbrace{\frac{dx}{dt}}_{u} = \underbrace{+\frac{dX}{dt}\cos\Omega t + \frac{dY}{dt}\sin\Omega t}_{U} \overbrace{-\Omega X \sin\Omega t + \Omega Y \cos\Omega t}^{+\Omega y}$$

$$\underbrace{\frac{dy}{dt}}_{v} = \underbrace{-\frac{dX}{dt}\sin\Omega t + \frac{dY}{dt}\cos\Omega t}_{V} \underbrace{-\Omega X \cos\Omega t - \Omega Y \sin\Omega t}_{-\Omega x}$$

Relation between absolute and relative velocity

$$U = u - \Omega y, \qquad V = v + \Omega x.$$

absolute velocity = relative velocity + entraining velocity due to rotation

A rotating framework - *The acceleration (2nd derivative)*

$$\underbrace{\frac{d^2x}{dt^2}}_{\frac{du}{dt}} = \underbrace{\left(\frac{d^2X}{dt^2}\cos\Omega t + \frac{d^2Y}{dt^2}\sin\Omega t\right)}_{\frac{dU}{dt}} + \underbrace{2\Omega\underbrace{\left(-\frac{dX}{dt}\sin\Omega t + \frac{dY}{dt}\cos\Omega t\right)}_{V}}_{2\Omega V}$$

$$\underbrace{-\quad\Omega^2\underbrace{\left(X\ \cos\Omega t\ +\ Y\ \sin\Omega t\right)}_{x}}_{-\Omega^2 x}$$

$$\underbrace{\frac{d^2y}{dt^2}}_{\frac{dv}{dt}} = \underbrace{\left(-\frac{d^2X}{dt^2}\sin\Omega t + \frac{d^2Y}{dt^2}\cos\Omega t\right)}_{\frac{dV}{dt}} \underbrace{-\ 2\Omega\underbrace{\left(\frac{dX}{dt}\cos\Omega t + \frac{dY}{dt}\sin\Omega t\right)}_{U}}_{-2\Omega U}$$

$$\underbrace{-\quad\Omega^2\underbrace{\left(-\ X\ \sin\Omega t\ +\ Y\ \cos\Omega t\right)}_{y}}_{-\Omega^2 y}.$$

A rotating framework - *The acceleration (2nd derivative)*

$$\frac{du}{dt} \qquad \frac{dU}{dt} \qquad 2\Omega V$$

$$\frac{d^2x}{dt^2} = \left(\frac{d^2X}{dt^2}\cos\Omega t + \frac{d^2Y}{dt^2}\sin\Omega t\right) + 2\Omega\underbrace{\left(-\frac{dX}{dt}\sin\Omega t + \frac{dY}{dt}\cos\Omega t\right)}_{V}$$

$$- \quad \Omega^2\underbrace{\left(X\ \cos\Omega t\ +\ Y\ \sin\Omega t\right)}_{x}$$

$$-\Omega^2 x$$

Relation between absolute and relative velocity

use this equality:

$$U = u - \Omega y, \qquad V = v + \Omega x.$$

A rotating framework - *The acceleration (2nd derivative)*

$$\frac{dU}{dt} = \frac{du}{dt} - 2\Omega v - \Omega^2 x$$

$$\frac{dV}{dt} = \frac{dv}{dt} - 2\Omega u - \Omega^2 y$$

absolute acceleration	=	relative acceleration	+	Coriolis acceleration	+	Centrifugal acceleration

use this equality:

$$U = u - \Omega y, \qquad V = v + \Omega x.$$

A rotating framework - *The acceleration (2nd derivative)*

define vectors: $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}$

Vector Notation

$$\frac{d\mathbf{u}}{dt} + \boxed{2\boldsymbol{\Omega} \times \mathbf{u}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

Coriolis acceleration

Centrifugal acceleration

time derivative in rotating frame:

$$\frac{d}{dt} + \boldsymbol{\Omega} \times$$

Coriolis vs. Centrifugal Force

$$\frac{dU}{dt} = \frac{du}{dt} - 2\Omega v - \Omega^2 x$$

$$\frac{dV}{dt} = \frac{dv}{dt} - 2\Omega u - \Omega^2 y$$

Centrifugal Force depends only on location

Coriolis Force is active only when things move

Centrifugal Force is unimportant for motions

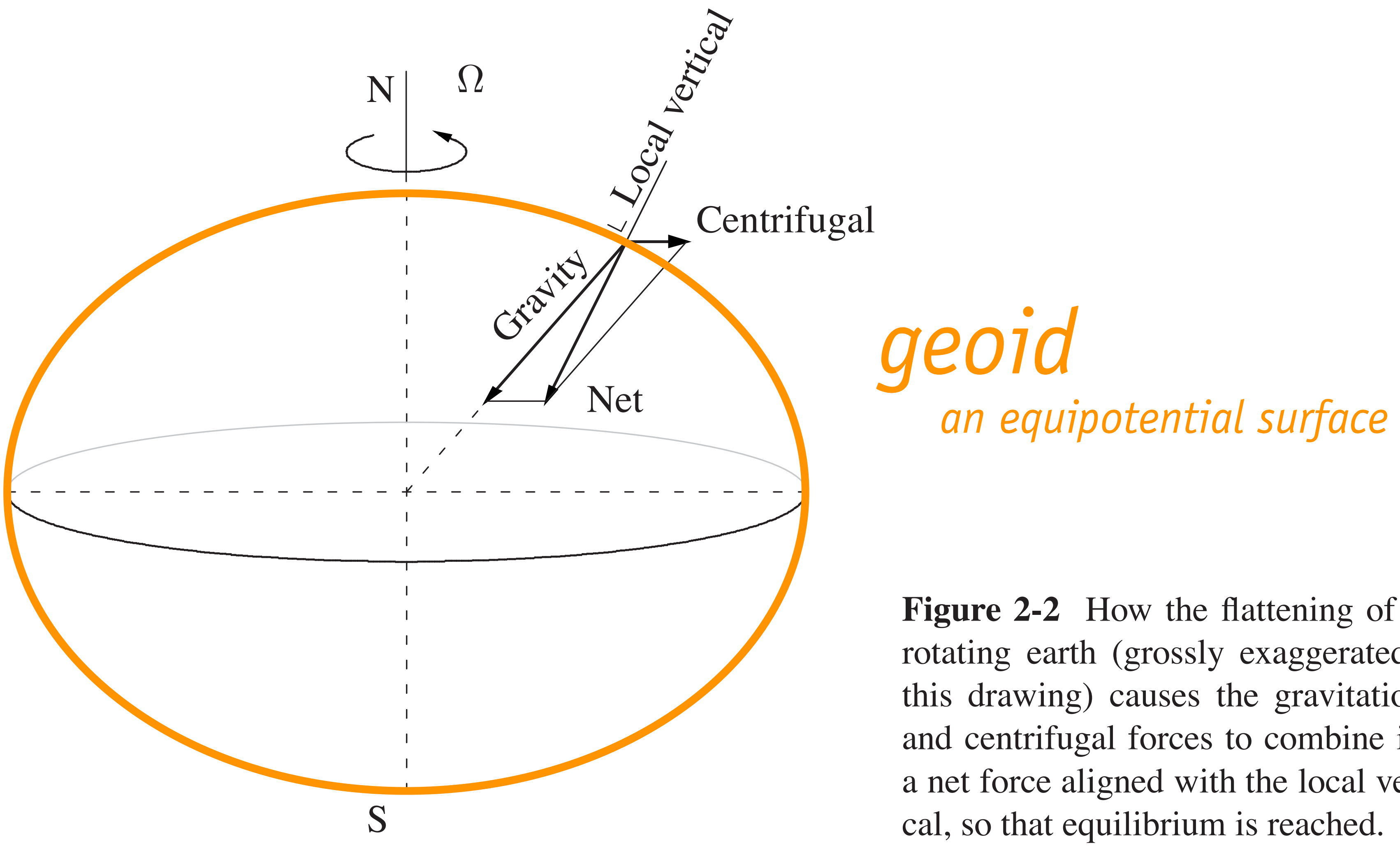


Figure 2-2 How the flattening of the rotating earth (grossly exaggerated in this drawing) causes the gravitational and centrifugal forces to combine into a net force aligned with the local vertical, so that equilibrium is reached.

Free Motion on a rotating frame

$$\frac{dU}{dt} = \frac{du}{dt} - 2\Omega v$$

$$\frac{dV}{dt} = \frac{dv}{dt} - 2\Omega u$$

Coriolis Force is active only when things move

Free Motion on a rotating frame

$$\frac{du}{dt} - 2\Omega v = 0, \quad \frac{dv}{dt} + 2\Omega u = 0.$$

The general solution to this system of linear equations is

$$\boxed{u = V \sin(ft + \phi), \quad v = V \cos(ft + \phi)}$$

$$f = 2\Omega$$

Inertial Oscillations

NOTE: the speed does not change with time
yet *u* and *v* do change with time!

↓

changes in *u* and *v* imply change in direction.

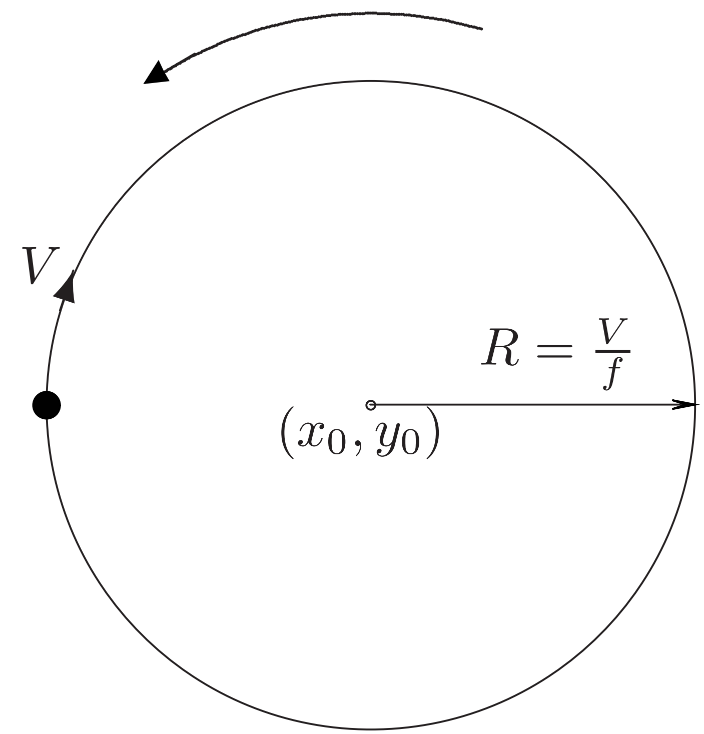
Trajectory of inertial oscillations

$$x = x_0 - \frac{V}{f}\cos(ft + \phi)$$

$$y = y_0 + \frac{V}{f}\sin(ft + \phi),$$

combine and take the square

$$(x - x_0)^2 + (y - y_0)^2 = \left(\frac{V}{f}\right)^2$$



1

this is the equation of a circle with radius

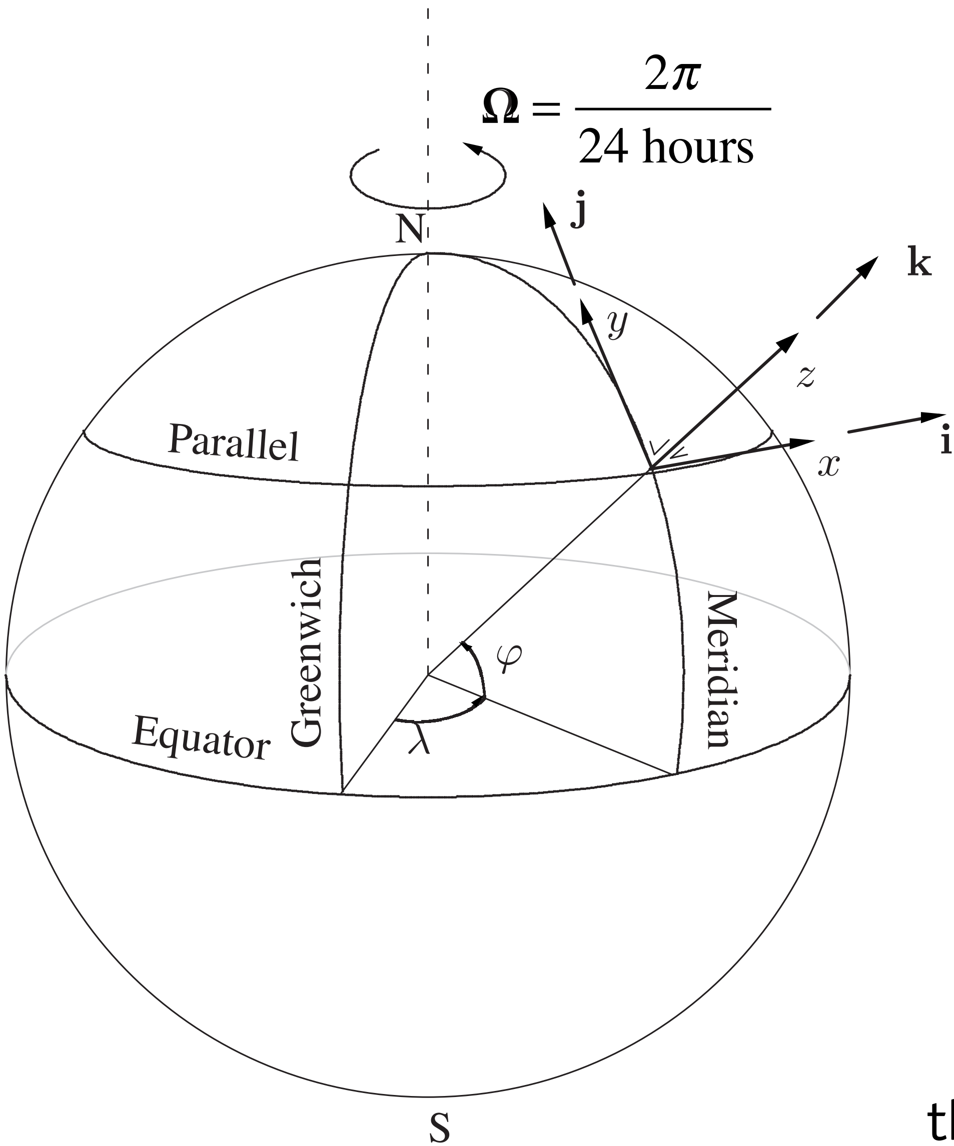
$$R = \frac{V}{f}$$

2

period of a complete circle is called ***inertial period***

$$T_p = 2\pi / f$$

Coriolis acceleration in 3D



effective rotation on the sphere

$$f = 2\Omega \sin\varphi$$

rotation of reference frame (2Ω)

projection on sphere surface ($\sin\varphi$)

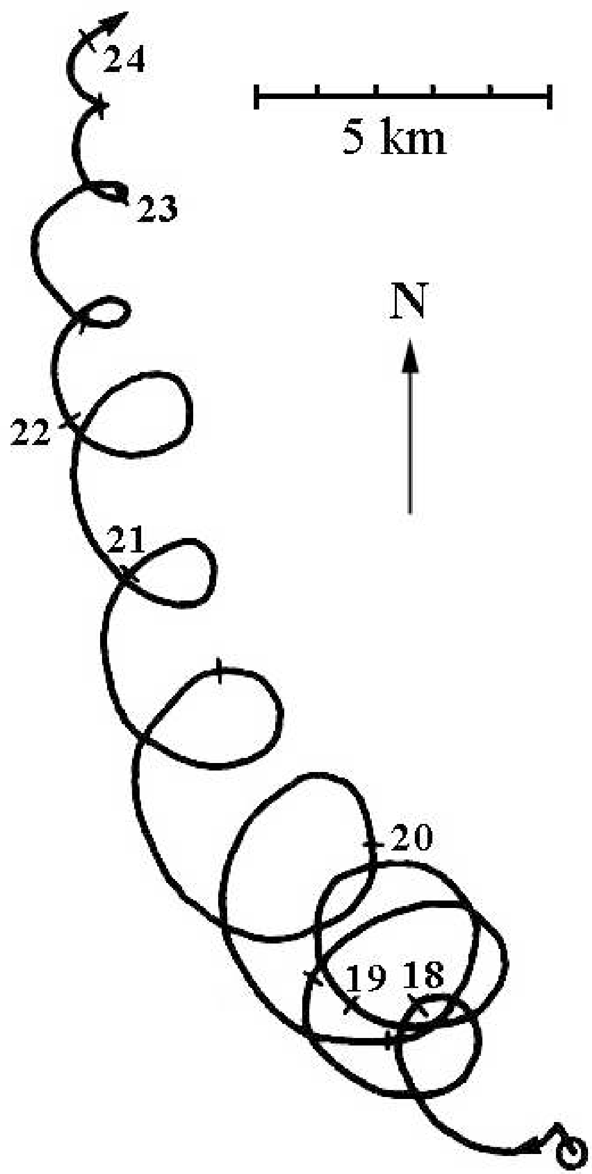
equation of inertial oscillations

$$\frac{du}{dt} - fv = 0$$

$$\frac{dv}{dt} + fu = 0$$

these describe the unforced motion

Coriolis acceleration in 3D - *observations of inertial motions*



equation of inertial oscillations

$$\frac{du}{dt} - fv = 0$$
$$\frac{dv}{dt} + fu = 0$$

these describe the unforced motion

Discretizing the intertial oscillation equations

Euler Method

$$\frac{du}{dt} - fv = 0 \longrightarrow \boxed{\frac{\tilde{u}^{n+1} - \tilde{u}^{n}}{\Delta t}} - f\tilde{v}^{n} = 0$$

$$\frac{dv}{dt} + fu = 0 \longrightarrow \boxed{\frac{\tilde{v}^{n+1} - \tilde{v}^{n}}{\Delta t}} + f\tilde{u}^{n} = 0$$

Euler Method Implicit

$$\frac{\tilde{u}^{n+1} - \tilde{u}^{n}}{\Delta t} - \boxed{f\tilde{v}^{n+1}} = 0$$

$$\frac{\tilde{v}^{n+1} - \tilde{v}^{n}}{\Delta t} + \boxed{f\tilde{u}^{n+1}} = 0$$

when rigth-hand side is at future time