III Stratification Effects

11 Stratification

11.1 Introduction ................................................................................................................. 319
11.2 Static stability ................................................................................................................ 320
11.3 A note on atmospheric stratification ........................................................................... 321
11.4 Convective adjustment ................................................................................................. 326
11.5 The importance of stratification: The Froude number ............................................... 327
11.6 Combination of rotation and stratification ................................................................. 330

Analytical Problems ........................................................................................................ 332
Numerical Exercises .......................................................................................................... 333
Biography: David Brunt ..................................................................................................... 334
Biography: Vilho Väisälä ................................................................................................. 335

12 Layered Models

12.1 From depth to density ................................................................................................. 337
12.2 Layered models ........................................................................................................... 340
12.3 Potential vorticity ....................................................................................................... 346
12.4 Two-layer models ....................................................................................................... 347
12.5 Wind-induced seiches and resonance in lakes ............................................................. 351
12.6 Energy conservation .................................................................................................... 352
12.7 Numerical layered models .......................................................................................... 354
12.8 Lagrangian approach .................................................................................................. 360

Analytical Problems ........................................................................................................ 362
Numerical Exercises .......................................................................................................... 363
Biography: Raymond Braislin Montgomery ................................................................. 365
Biography: Jörg Imberger ............................................................................................... 366
11.2 Static Stability

Figure 11-1 When an incompressible fluid parcel of density $\rho(z)$ is vertically displaced from level $z$ to level $z + h$ in a stratified environment, a buoyancy force appears because of the density difference $\rho(z) - \rho(z + h)$ between the particle and the ambient fluid.

Displace a particle upward $z + h$

The different in weight generates a force, which generates acceleration

\[
g [\rho(z) - \rho(z + h)] V.
\]

Newton’s Law

\[
\rho(z) V \frac{d^2 h}{dt^2} = g [\rho(z + h) - \rho(z)] V.
\]

Boussinesq approximation allows us to replace rho(z) with rho0

Taylor expand

\[
\rho(z + h) - \rho(z) \approx \frac{d\rho}{dz} h.
\]

Derive

\[
\frac{d^2 h}{dt^2} - \frac{g}{\rho_0} \frac{d\rho}{dz} h = 0
\]

Stratification frequency

Brunt Vaisala

\[
N^2 = -\frac{g}{\rho_0} \frac{d\rho}{dz}
\]

If the coefficient in equation (11.1) is negative (i.e., $d\rho/dz > 0$).
11.4 Convective adjustment

When statically unstable in models

\[ N^2 \leq 0 \]

sample code for convective adjustment

```plaintext
while there is any denser fluid being on top of lighter fluid
  loop over all layers
    if density of layer above > density of layer below
      mix properties of both layers, with a volume-weighted average
    end if
  end loop over all layers
end while
```

is too strong in practice,

Graphical example of the code

![Graphical example of the code](image)

**Figure 11-4** Illustration of convective adjustment while the fluid is heated from below. Grid boxes below heavier neighbors are systematically mixed in pairs until the whole fluid column is rendered stable.

Therefore, numerical mixing should preferably be replaced
11.5 The importance of stratification: The Froude number

**Rossby Number**  \[
Ro = \frac{U}{\Omega L}
\]

we have derived a scaling for when rotation is important

Now derive scaling for when stratification is important!

> Consider the following:
> depth of fluid
> stratification N2
> speed U
> over obstacle of length L
> providing vertical displacement Dz

time spent over the obstacle \(T = \frac{L}{U}\)
to go over obstacle you need \(w = \frac{Dz}{T} = \frac{UDz}{L}\)
vertical displacement will cause a density perturbation of order

\[
\Delta \rho = \left| \frac{d\rho}{dz} \right| \Delta z \\
= \frac{\rho_0 N^2}{g} \Delta z,
\]

this density perturbation --> pressure disturbance
\[ \Delta P = gH \Delta \rho = \rho_0 N^2 H \Delta z. \]

The change in pressure leads to change in velocity

By virtue of the balance of forces in the horizontal, the pressure-gradient force must be accompanied by a change in fluid velocity \([u \partial u / \partial x + v \partial u / \partial y \sim (1/\rho_0) \partial p / \partial x]\):

\[ \frac{U^2}{L} = \frac{\Delta P}{\rho_0 L} \implies U^2 = N^2 H \Delta z. \quad (11.16) \]

We can estimate the rate of vertical convergence \((W/H)\) versus horizontal divergence \((U/L)\)

\[ \frac{W/H}{U/L} = \frac{\Delta z}{H} = \frac{U^2}{N^2 H^2}. \]

If \(U^2\) is less than \(N^2 H^2\), then the rate of vertical convergence cannot keep up with the flow.

We conclude that

\[ Fr = \frac{U}{NH}, \]

called the Froude number, is a measure of the importance of stratification. The rule is: If \(Fr \ll 1\), stratification effects are important; the smaller \(Fr\), the more important these effects are.
Compare Rossby and Froude number

\[ Ro = \frac{U}{\Omega L}, \quad Fr = \frac{U}{NH}, \quad \text{velocity / frequency* length scale} \]

Fr measure vertical velocity in stratified fluids, Ro vertical velocity in Rotating fluids

equivalent so that the term \( Fr \) measure vertical velocity in stratified fluids, \( Ro \) vertical velocity in Rotating fluids.

example of fluid going around obstacles in the horizontal!

when \( Fr \) and \( Ro \ll 1 \)
11.6 Combination of rotation and stratification

\[ \Omega U = \frac{\Delta P}{\rho_0 L} \implies U = \frac{N^2 H \Delta z}{\Omega L} \]

Geostrophic balance

replace Dp with \[ \Delta P = gH \Delta \rho = \rho_0 N^2 H \Delta z. \]

\[
\frac{W/H}{U/L} = \frac{\Delta z}{H} = \frac{\Omega LU}{N^2 H^2} = \frac{Fr^2}{Ro}.
\]

Flow influenced mostly by rotation

Three-dimensional flow

\[ W/H \sim Fr^2 \]

\[ W/H \sim Fr \]

\[ W/H \sim Ro \]

Flow influenced by both rotation and stratification

Flow influenced mostly by stratification

When Fr and Ro are same order

\[ Ro = \frac{U}{\Omega L}, \quad Fr = \frac{U}{NH}, \quad L = \frac{NH}{\Omega}. \]
Layered Models

density increasing with depth we can use as coordinate and model fluid on isopycnal surface

\[ \frac{\partial}{\partial x} \rightarrow \frac{\partial a}{\partial x} \bigg|_z = \frac{\partial a}{\partial x} \bigg|_\rho + \frac{\partial a}{\partial \rho} \frac{\partial \rho}{\partial x} \bigg|_z \]
\[ \frac{\partial}{\partial y} \rightarrow \frac{\partial a}{\partial y} \bigg|_z = \frac{\partial a}{\partial y} \bigg|_\rho + \frac{\partial a}{\partial \rho} \frac{\partial \rho}{\partial y} \bigg|_z \]
\[ \frac{\partial}{\partial z} \rightarrow \frac{\partial a}{\partial z} = \frac{\partial a}{\partial \rho} \frac{\partial \rho}{\partial z} \]
\[ \frac{\partial}{\partial t} \rightarrow \frac{\partial a}{\partial t} \bigg|_z = \frac{\partial a}{\partial t} \bigg|_\rho + \frac{\partial a}{\partial \rho} \frac{\partial \rho}{\partial t} \bigg|_z \]

**Figure 12-1** Geometrical interpretation of equation (12.1). The \(x\)-derivatives of any function \(a\) at constant depth \(z\) and at constant density \(\rho\) are, respectively, \([a(B) - a(A)]/\Delta x\) and \([a(C) - a(A)]/\Delta z\). The difference between the two, \([a(C) - a(B)]/\Delta x\), represents the vertical derivative of \(a\), \([a(C) - a(B)]/\Delta z\), times the slope of the density surface, \(\Delta z/\Delta x\). Finally, the vertical derivative can be split as the ratio of the \(\rho\)-derivative of \(a\), \([a(C) - a(B)]/\Delta \rho\), by \(\Delta z/\Delta \rho\).
Used text from Book chapter 12 for derivation of equations:

\[
\begin{align*}
\frac{du}{dt} - fu &= -\frac{1}{\rho_0} \frac{\partial P}{\partial x} \\
\frac{dv}{dt} + fu &= -\frac{1}{\rho_0} \frac{\partial P}{\partial y} \\
\frac{\partial P}{\partial \rho} &= gz, \quad P = p + \rho gz \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0 \\
h &= -\Delta \rho \frac{\partial z}{\partial \rho} \\
g' &= \frac{\Delta \rho}{\rho_0} g,
\end{align*}
\]
12.3 Potential vorticity

For layered models, we can reproduce the vorticity analysis that we performed on the shallow-water model (Section 7.4). First, the relative vorticity $\zeta$ of the flow at any level is defined as

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad (12.20)$$

and the expression for potential vorticity is defined in analogy with (7.28):

$$q = \frac{f + \zeta}{h} = \frac{f + \partial v/\partial x - \partial u/\partial y}{h}, \quad (12.21)$$

which is identical to the expression for a barotropic fluid, except that the denominator is now a differential thickness given by (12.10) rather than the full thickness of the system. It can be shown that in the absence of friction, expression (12.21) is conserved by the flow (its material derivative is zero).

The interpretation of this conservation property follows that for a barotropic fluid: When the fluid layer between two consecutive density surface is squeezed (from left to right in Figure 12-4), conservation of volume demands that it widens, and conservation of circulation in turn requires that it spins less fast; the net effect is that the vorticity $f + \zeta$ decreases in proportion to the thickness $h$ of the fluid layer.

![Figure 12-4](image-url) Conservation of volume and circulation in a fluid undergoing divergence (squeezing) or convergence (stretching). The products of $h\, ds$ and $(f + \zeta)\, ds$ are conserved during the transformation, implying conservation of $(f + \zeta)/h$, too.