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11.2 Static Stability

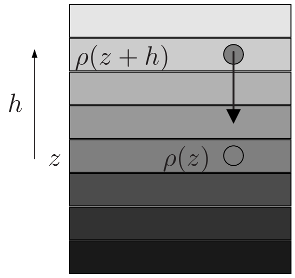


Figure 11-1 When an incompressible fluid parcel of density $\rho(z)$ is vertically displaced from level z to level $z+h$ in a stratified environment, a buoyancy force appears because of the density difference $\rho(z) - \rho(z+h)$ between the particle and the ambient fluid.

displace a particle upward $z + h$

the different in weight generates a force, which generates and acceleration

$$g\,[\rho(z)\ -\ \rho(z+h)]\,V,$$

Newton's Law

$$\rho(z)\,V\,\frac{d^2h}{dt^2}\ =\ g\,[\rho(z+h)\ -\ \rho(z)]\,V.$$

Boussinesq approximation allows us to replace rho(z) with rho0

Taylor expand

$$\rho(z+h)\ -\ \rho(z)\ \simeq\ \frac{d\rho}{dz}\,h.$$

Derive

$$\frac{d^2h}{dt^2}\ \boxed{-\ \frac{g}{\rho_0}\,\frac{d\rho}{dz}}\,h\ =\ 0$$

Stratification frequency
Brunt Vaisala

$$\boxed{N^2\ =\ -\ \frac{g}{\rho_0}\,\frac{d\rho}{dz}}$$

If the coefficient in equation (11.1) is negative ($i.e.$, $d\rho/dz > 0$.

11.4 Convective adjustment

When statically unstable in models $N^2 \leq 0$

sample code for convective adjustment

```
while there is any denser fluid being on top of lighter fluid
   loop over all layers
      if density of layer above > density of layer below
         mix properties of both layers, with a volume-weighted average
      end if
   end loop over all layers
end while
```

is too strong in practice,

Graphical example of the code

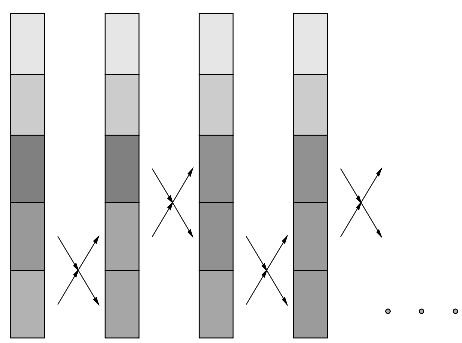


Figure 11-4 Illustration of convective adjustment while the fluid is heated from below. Grid boxes below heavier neighbors are systematically mixed in pairs until the whole fluid column is rendered stable.

Therefore, numerical mixing should preferably be replaced

11.5 The importance of stratification: The Froude number

Rossby Number
advection/rotation

$$Ro = \frac{U}{\Omega L}$$

we have derived a scaling for when rotation is important

Now derive scaling for when stratification is important!

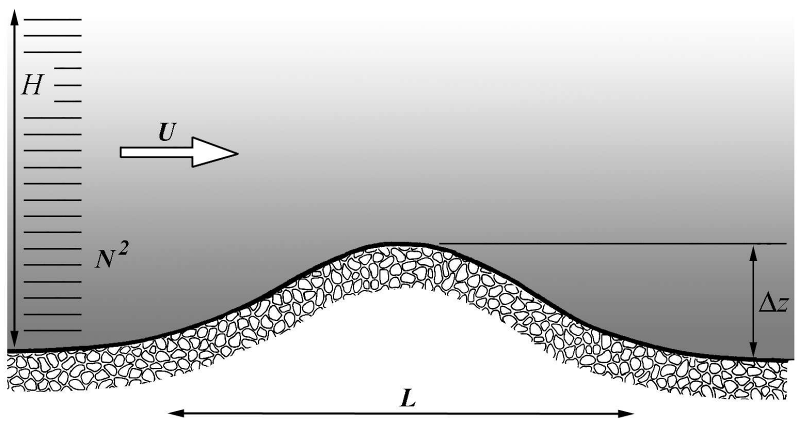


Figure 11-5 Situation in which a stratified flow encounters an obstacle, forcing some fluid parcels to move vertically against a buoyancy force.

Consider the following:
depth of fluid
stratification N2
speed U
over obstacle of length L
providing vertical displacement Dz

time spent over the obstacle T= L/U
to go over obstacle you need w=Dz/T = UDz/L
vertical displacement will cause a density perturbation of order

$$\begin{aligned}\Delta\rho &= \left|\frac{d\bar{\rho}}{dz}\right| \Delta z \\ &= \frac{\rho_0 N^2}{g} \Delta z,\end{aligned}$$

this density perturbation --> pressure disturbance

$$\Delta P = gH\Delta\rho$$
$$= \rho_0 N^2 H \Delta z.$$

the change in pressure leads to change in velocity

By virtue of the balance of forces in the horizontal, the pressure-gradient force must be accompanied by a change in fluid velocity $[u\partial u/\partial x + v\partial u/\partial y \sim (1/\rho_0)\partial p/\partial x]$:

$$\frac{U^2}{L} = \frac{\Delta P}{\rho_0 L} \implies U^2 = N^2 H \Delta z. \tag{11.16}$$

we can estimate the rate of vertical convergence (W/H) versus horizontal divergence (U/L)

$$\frac{W/H}{U/L} = \frac{\Delta z}{H} = \frac{U^2}{N^2 H^2}.$$

if U^2 is less than N2H2, than the rate of vertical convergence cannot keep up with the flow

we conclude that

$$Fr = \frac{U}{NH},$$

called the *Froude number*, is a measure of the importance of stratification. The rule is: If $Fr \lesssim 1$, stratification effects are important; the smaller Fr, the more important these effects are.

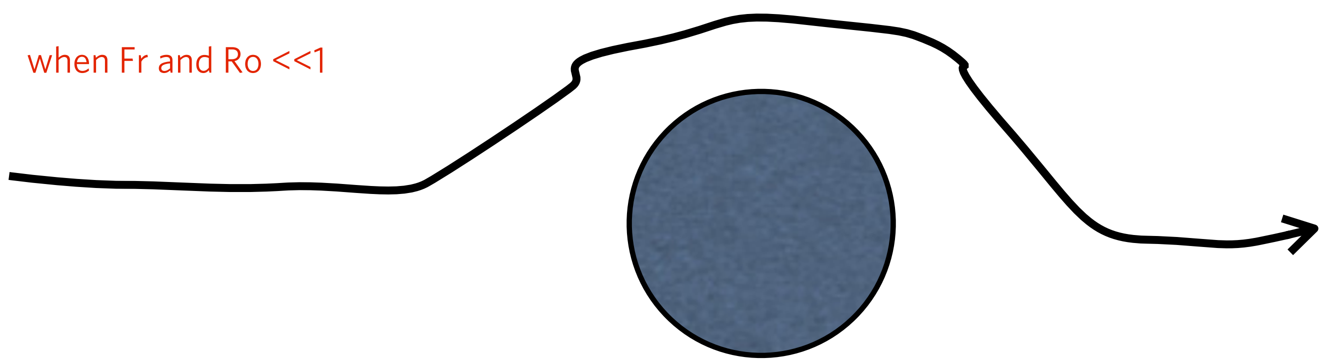
Compare Rossby and Froude number

$$Ro = \frac{U}{\Omega L}; \qquad Fr = \frac{U}{NH},$$ velocity / frequency* length scale

Fr measure vertical velocity in stratified fluids, Ro vertical velocity in Rotating fluids

example of fluid going around obstacles in the horizontal!

when Fr and Ro <<1



11.6 Combination of rotation and stratification

$$\Omega U = \frac{\Delta P}{\rho_0 L} \implies U = \frac{N^2 H \Delta z}{\Omega L}$$

Geostrophic balance
replace Dp with

$$\Delta P = gH\Delta\rho = \rho_0 N^2 H \Delta z.$$

$$\frac{W/H}{U/L} = \frac{\Delta z}{H} = \frac{\Omega L U}{N^2 H^2} = \frac{Fr^2}{Ro}.$$

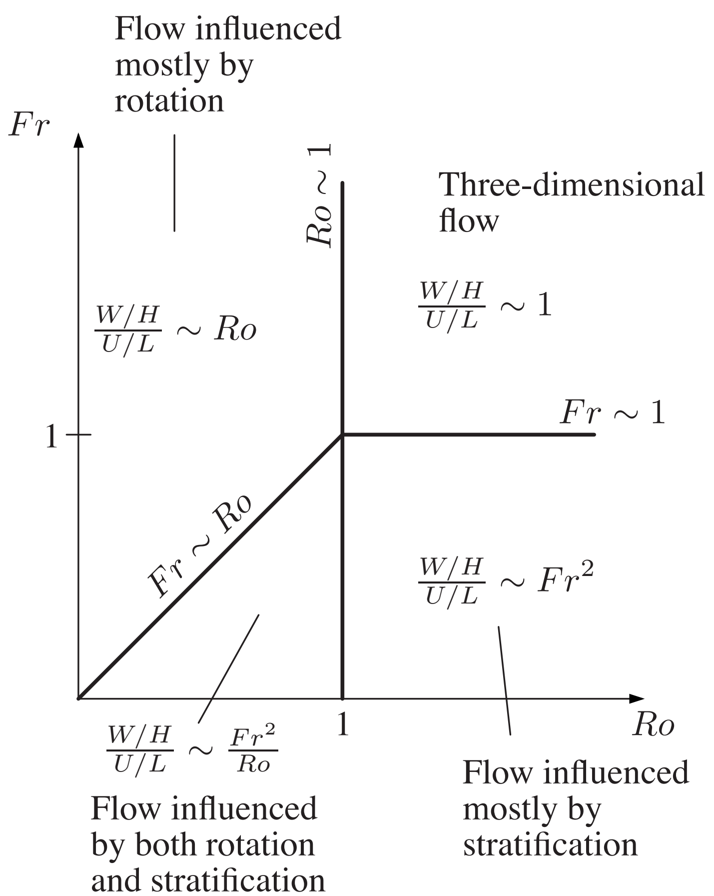


Figure 11-6 Recapitulation of the various scalings of the ratio of vertical convergence (divergence), W/H, to horizontal divergence (convergence), U/L, as a function of the Rossby number, $Ro = U/(\Omega L)$, and Froude number, $Fr = U/(NH)$.

internal radius of deformation.

When Fr and Ro are same order

$$Ro = \frac{U}{\Omega L}; \quad Fr = \frac{U}{NH}, \longrightarrow L = \frac{NH}{\Omega}.$$

Layered Models

density increasing with depth we can use as coordinate and model fluid on **isopycnal surface**

$$
\begin{aligned}
\frac{\partial}{\partial x} \quad &\longrightarrow \quad \left.\frac{\partial a}{\partial x}\right|_z = \left.\frac{\partial a}{\partial x}\right|_\rho + \frac{\partial a}{\partial \rho}\left.\frac{\partial \rho}{\partial x}\right|_z \\
\frac{\partial}{\partial y} \quad &\longrightarrow \quad \left.\frac{\partial a}{\partial y}\right|_z = \left.\frac{\partial a}{\partial y}\right|_\rho + \frac{\partial a}{\partial \rho}\left.\frac{\partial \rho}{\partial y}\right|_z \\
\frac{\partial}{\partial z} \quad &\longrightarrow \quad \frac{\partial a}{\partial z} = \frac{\partial a}{\partial \rho}\frac{\partial \rho}{\partial z} \\
\frac{\partial}{\partial t} \quad &\longrightarrow \quad \left.\frac{\partial a}{\partial t}\right|_z = \left.\frac{\partial a}{\partial t}\right|_\rho + \frac{\partial a}{\partial \rho}\left.\frac{\partial \rho}{\partial t}\right|_z .
\end{aligned}
$$

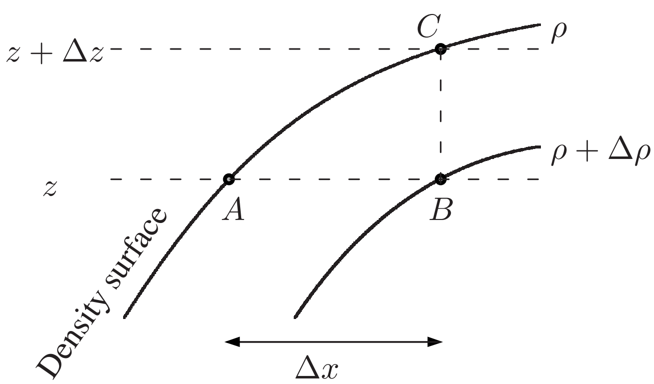


Figure 12-1 Geometrical interpretation of equation (12.1). The x-derivatives of any function a at constant depth z and at constant density ρ are, respectively, $[a(B) - a(A)]/\Delta x$ and $[a(C)-a(A)]/\Delta x$. The difference between the two, $[a(C) - a(B)]/\Delta x$, represents the vertical derivative of a, $[a(C) - a(B)]/\Delta z$, times the slope of the density surface, $\Delta z/\Delta x$. Finally, the vertical derivative can be split as the ratio of the ρ-derivative of a, $[a(C) - a(B)]/\Delta\rho$, by $\Delta z/\Delta\rho$.

Used text from Book chapter 12 for derivation of equations:

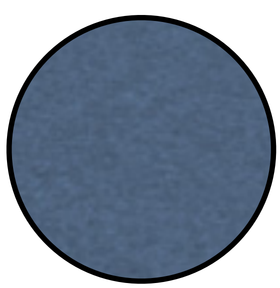
$$\frac{du}{dt} - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$
$$\frac{dv}{dt} + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}$$

$$\frac{\partial P}{\partial \rho} = gz, \qquad P = p + \rho g z$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

$$h = -\Delta\rho\frac{\partial z}{\partial \rho}$$

$$g' = \frac{\Delta\rho}{\rho_0}\, g,$$



12.3 Potential vorticity

For layered models, we can reproduce the vorticity analysis that we perfomed on the shallow-water model (Section 7.4). First, the relative vorticity ζ of the flow at any level is defined as

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{12.20}$$

and the expression for potential vorticity is defined in analogy with (7.28):

$$\begin{aligned} q &= \frac{f+\zeta}{h} \\ &= \frac{f + \partial v/\partial x - \partial u/\partial y}{h}, \end{aligned} \tag{12.21}$$

which is identical to the expression for a barotropic fluid, except that the denominator is now a differential thickness given by (12.10) rather than the full thickness of the system. It can be shown that in the absence of friction, expression (12.21) is conserved by the flow (its material derivative is zero).

The interpretation of this conservation property follows that for a barotropic fluid: When the fluid layer between two consecutive density surface is squeezed (from left to right in Figure 12-4), conservation of volume demands that it widens, and conservation of circulation in turn requires that it spins less fast; the net effect is that the vorticity $f + \zeta$ decreases in proportion to the thickness h of the fluid layer.

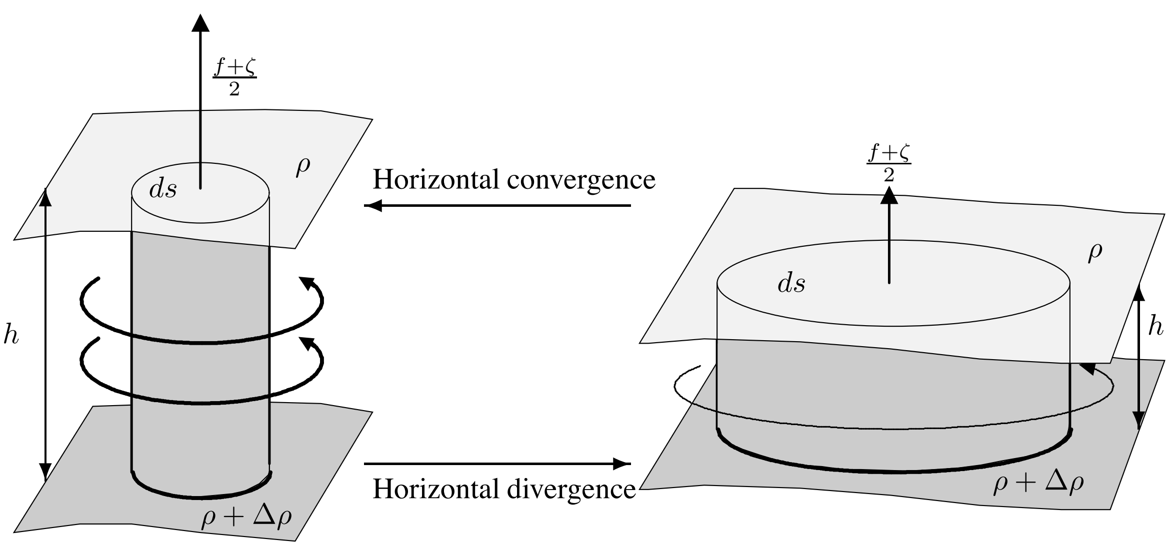


Figure 12-4 Conservation of volume and circulation in a fluid undergoing divergence (squeezing) or convergence (stretching). The products of $h\, ds$ and $(f+\zeta)\, ds$ are conserved during the transformation, implying conservation of $(f+\zeta)/h$, too.