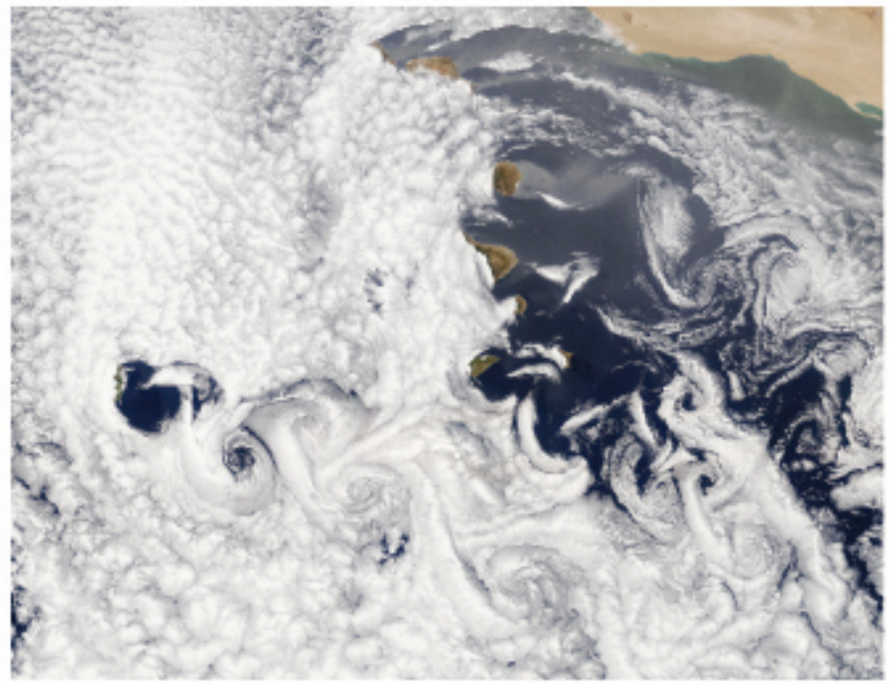
Ocean Modeling - EAS 8803

Boundary Conditions

- Type of boundary conditions (e.g. kinematic and dynamic)
- Numerical implementation
- Surface & bottom boundary condition for momentum
- The Ekman boundary layers

Introduction to
Geophysical Fluid Dynamics
Physical and Numerical Aspects



Benoit Cushman-Roisin and Jean-Marie Beckers

Academic Press

Chapter 4 and 8

Boundary Conditions

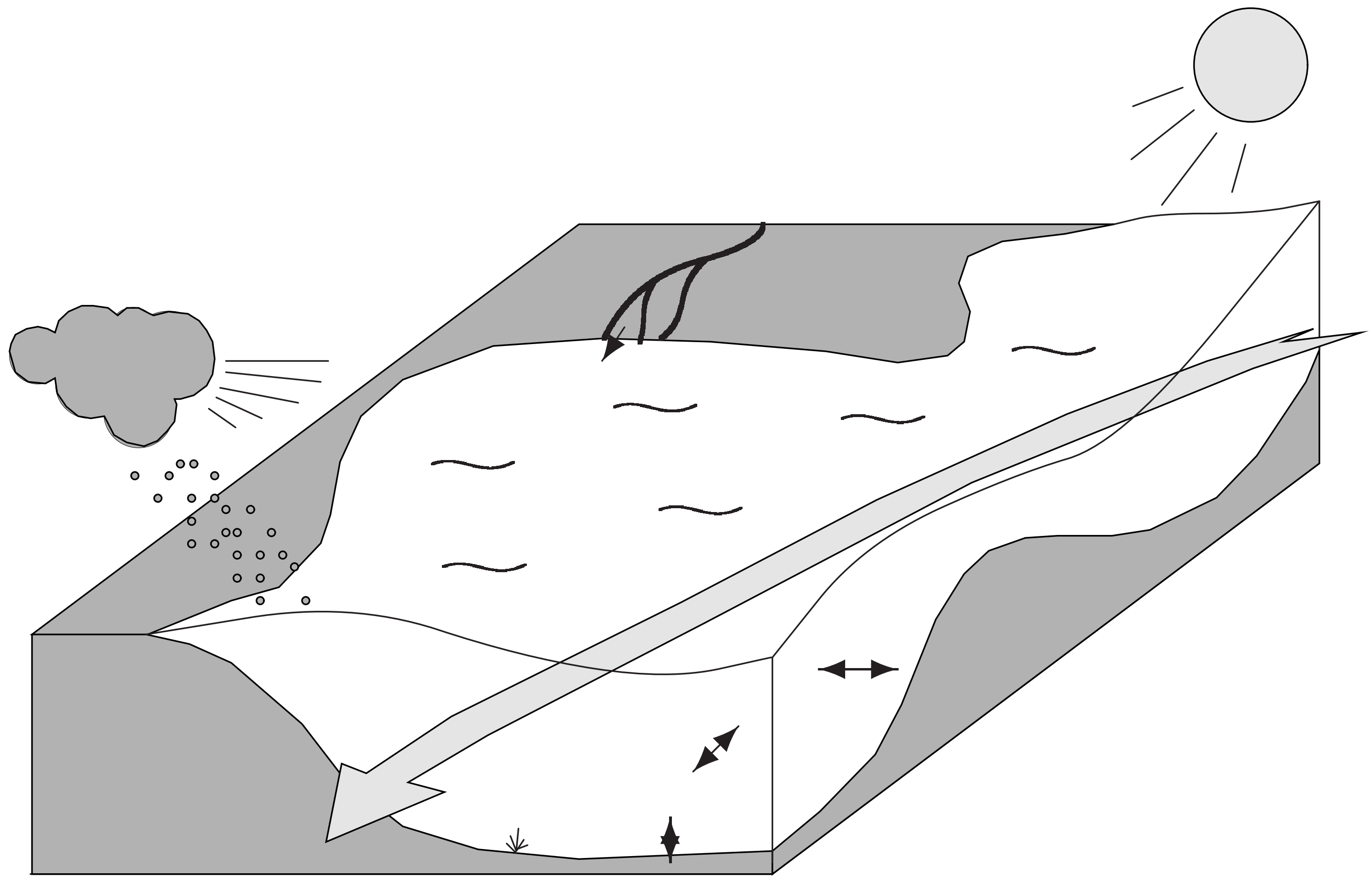


Figure 4-1 Schematic representation of possible exchanges between the system under investigation and the surrounding environment. Boundary conditions must specify the influence of this outside world on the evolution within the domain. Exchanges may take place at the air-sea interface, in bottom layers, along coasts and/or at any other boundary of the domain.

Boundary Conditions

state variables (variables with time derivative)

$$x - \text{momentum:} \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv =$$

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right)$$

$$y - \text{momentum:} \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu =$$

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial v}{\partial z}\right)$$

$$z - \text{momentum:} \quad 0 = -\frac{\partial p}{\partial z} - \rho g$$

$$\text{continuity:} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

diagnostic variables (variables with no time derivative)

$$\text{energy:} \quad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} =$$

$$\frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial \rho}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial \rho}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa_E\frac{\partial \rho}{\partial z}\right) ,$$

Kinematic Boundary Conditions

flow cannot penetrate solid boundaries (e.g. land, or bottom topography)

impermeability condition

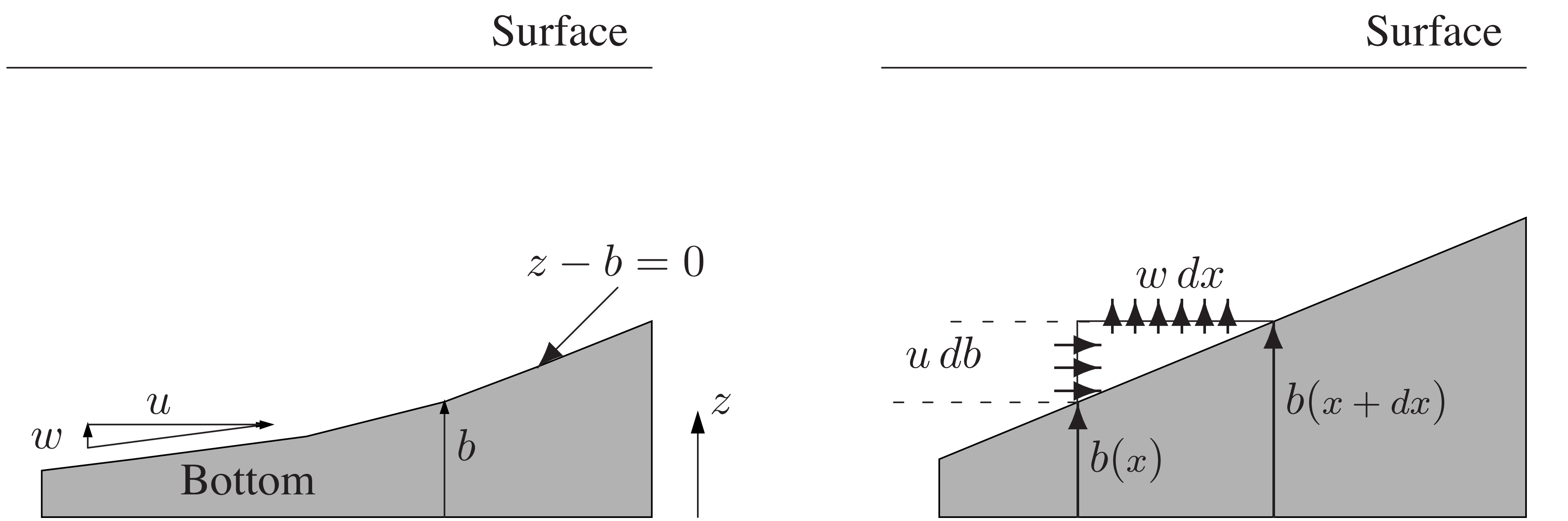


Figure 4-2 Notation and two physical interpretations of the bottom boundary condition illustrated here in a (x, z) plane for a topography independent of y. The impermeability of the bottom imposes that the velocity be tangent to the bottom defined by $z - b = 0$. In terms of the fluid budget, which can be extended to a finite volume approach, expressing that the horizontal inflow matches the vertical outflow requires $u\,(b(x + dx) - b(x)) = w\,dx$, which for $dx \to 0$ leads to (4.28). Note that the velocity ratio w/u is equal to the topographic slope db/dx, which scales like the ratio of vertical to horizontal length scales, *i.e.*, the aspect ratio.

Volume flux balance at bottom

$$w = u \frac{\partial b}{\partial x} + v \frac{\partial b}{\partial y}$$

Kinematic Boundary Conditions

flow cannot penetrate solid boundaries (e.g. land, or bottom topography)
impermeability condition

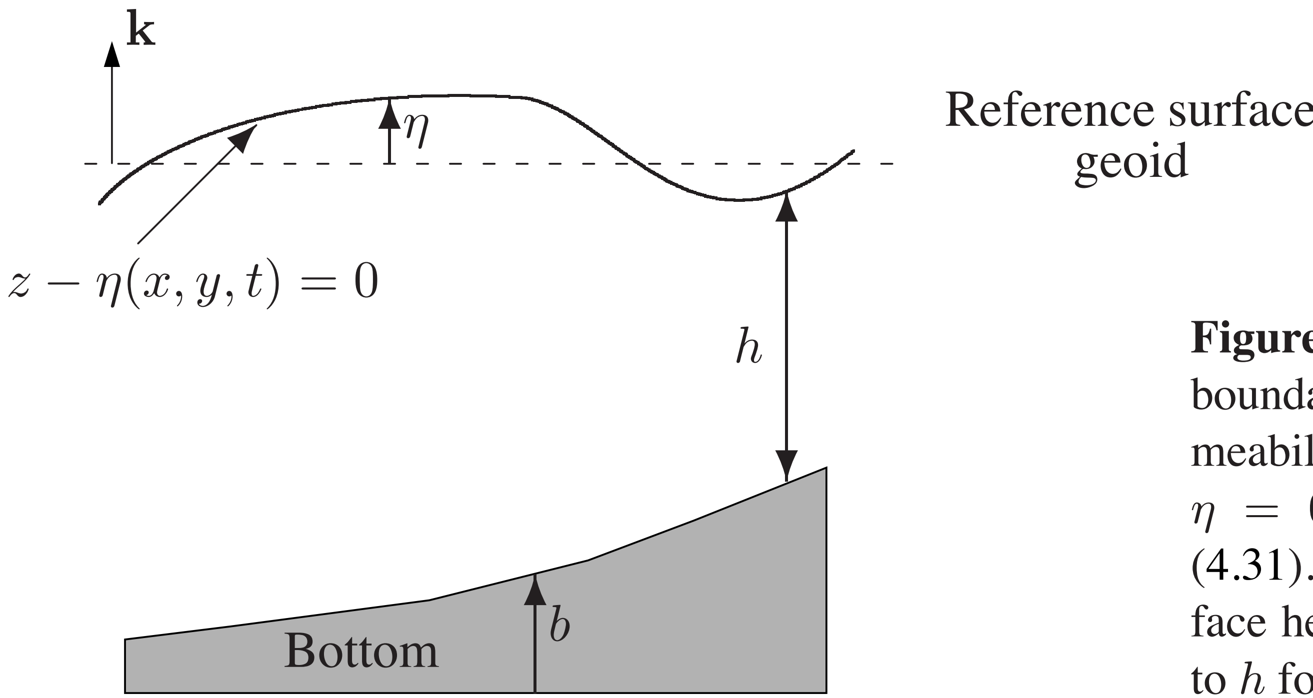


Figure 4-3 Notation for the surface boundary condition. Expressing impermeability of the moving surface $z - \eta = 0$ results in boundary condition (4.31). (The elevation of the sea surface height η is exaggerated compared to h for the purpose of illustration.)

Volume flux balance at surface

$$w = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} \quad \text{at} \quad z = \eta.$$

Kinematic Boundary Conditions

additional lateral boundary conditions for the free-surface

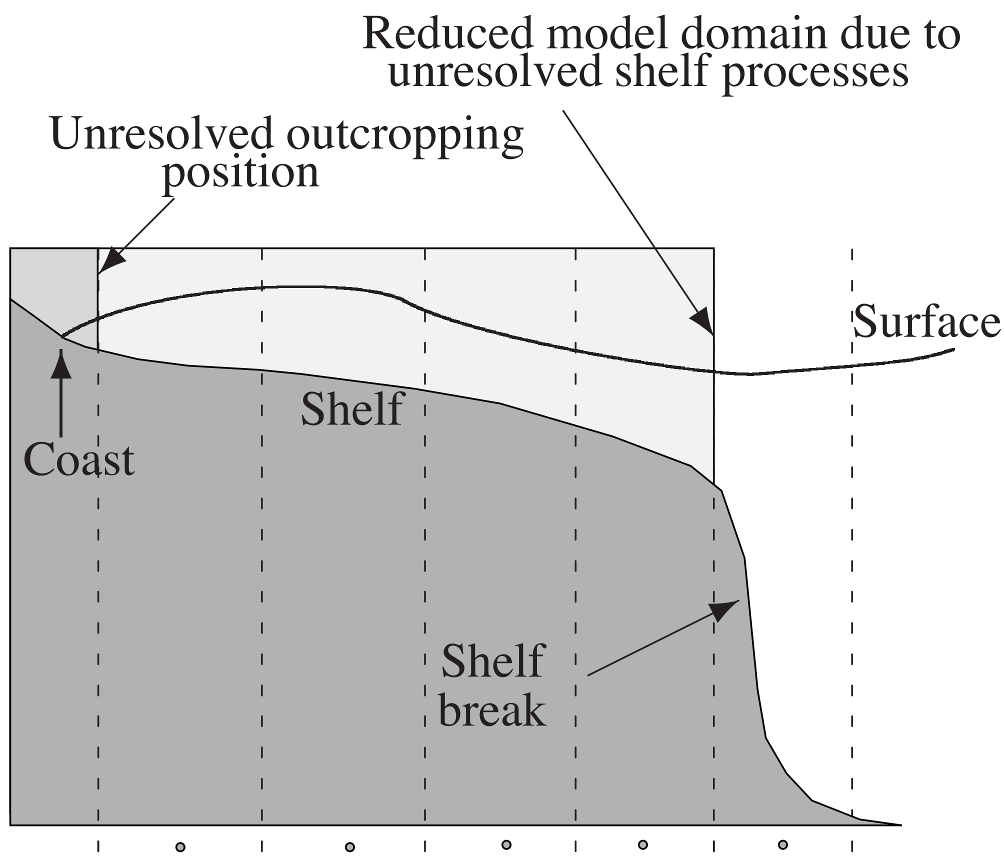


Figure 4-4 Vertical section across an oceanic domain reaching the coast. Besides surface and bottom boundaries, the coast introduces an additional *lateral* boundary. Introducing an artificial vertical wall is necessary because a fixed numerical grid cannot describe well the exact position of the water's edge. Occasionally, a vertical wall is assumed at the shelf break, removing the entire shelf area from the domain, because the reduced physics of the model are incapable of representing some processes on the shelf.

Dynamic Boundary Conditions

ensuring a dynamic continuity at the boundaries

$$p_{\text{atm}} = p_{\text{sea}} \quad \text{at air-sea interface.} \qquad z = \eta$$

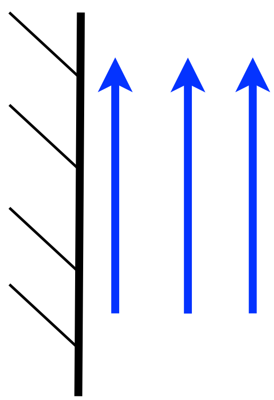
$$p_{\text{sea}}(z = 0) = p_{\text{atm at sea level}} + \rho_0 g \eta \qquad z = 0$$

Dynamic Boundary Conditions

ensuring a dynamic continuity at the boundaries

Free Slip

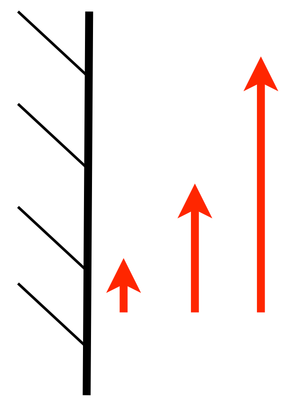
(friction at the lateral boundary is ignored)



impermeability kinematic condition

NO Slip

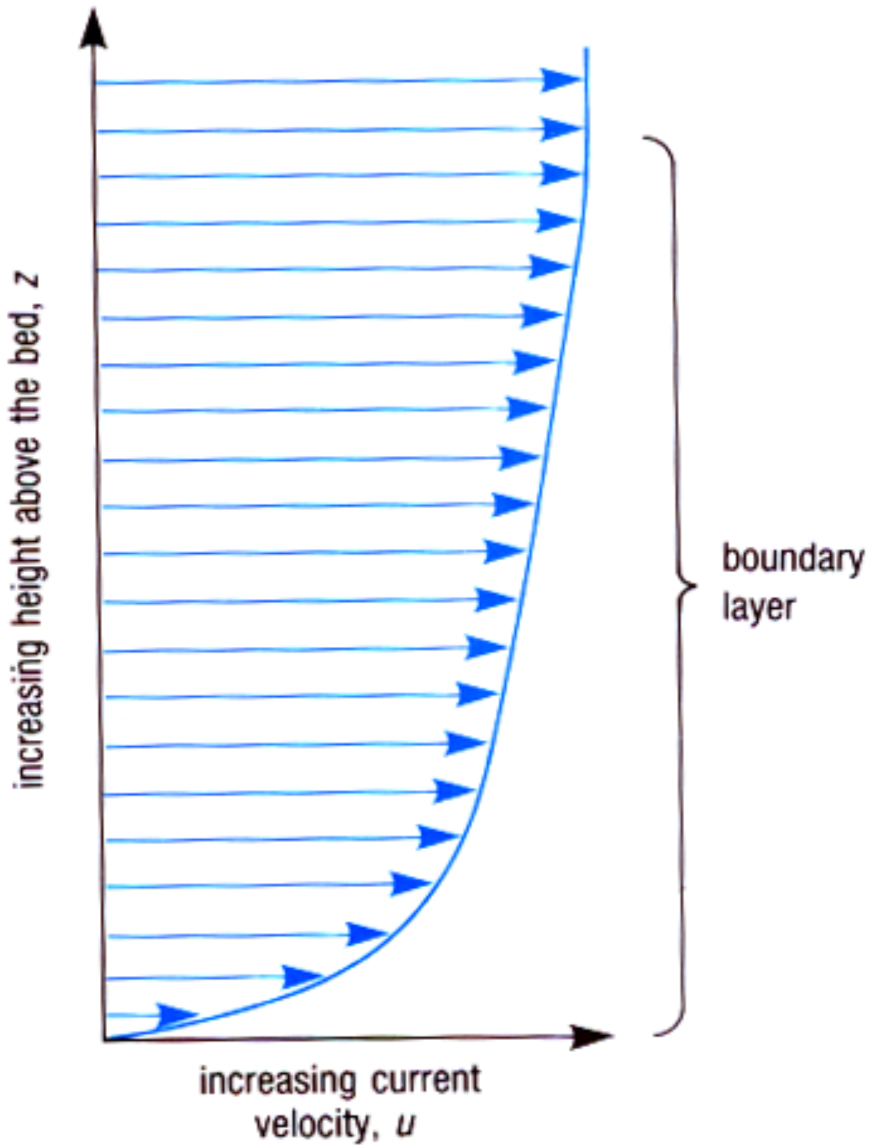
(friction at the lateral boundary is accounted for)



dynamic condition
velocity at boundary =0
--> Boundary Layer

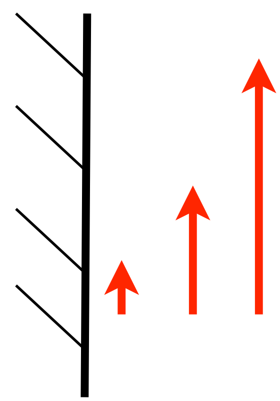
Dynamic Boundary Conditions

ensuring a dynamic continuity at the boundaries



NO Slip

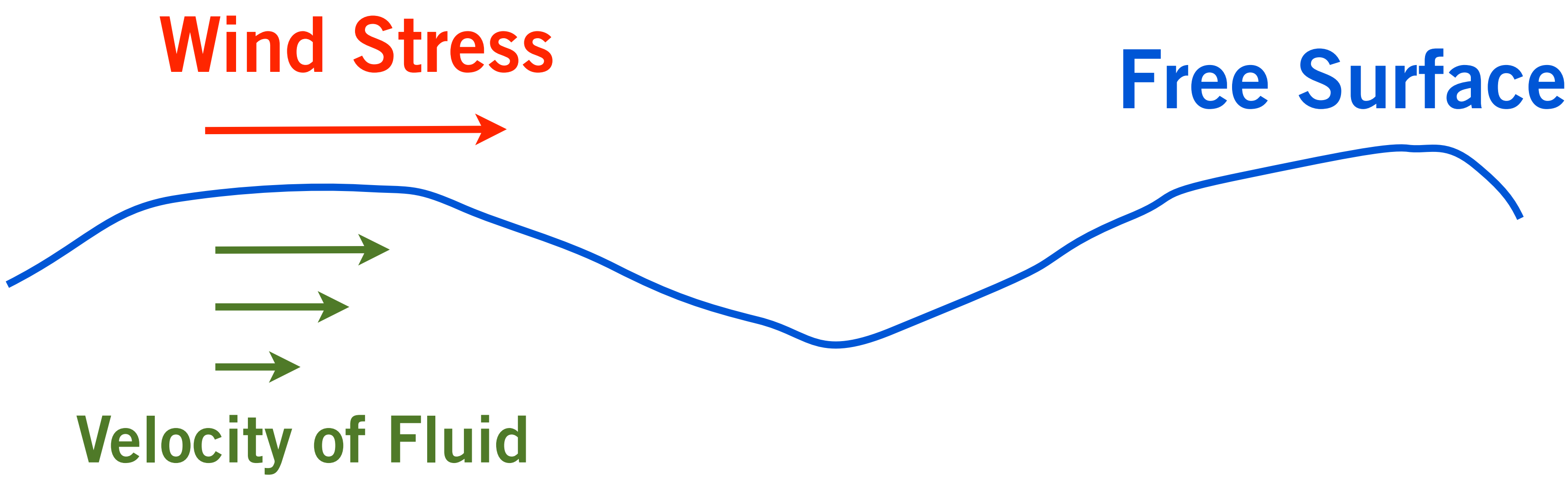
(friction at the lateral boundary is accounted for)



dynamic condition
velocity at boundary =0
--> Boundary Layer

Dynamic Boundary Conditions

ensuring a dynamic continuity at the boundaries, for a **moving boundary**



<u>Continuity of velocity and **tangential stresses**</u>

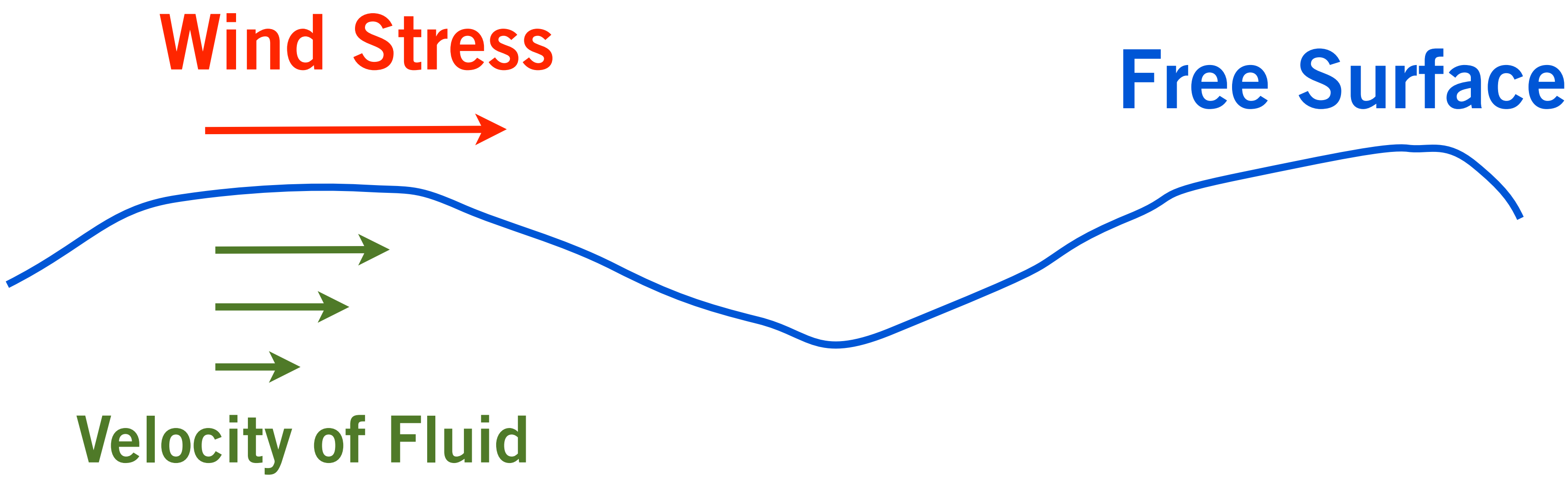
$$\rho_0 \nu_E \left(\frac{\partial v}{\partial z}\right)\Big|_{\text{at surface}} = \tau^y$$

$$\rho_0 \nu_E \left(\frac{\partial u}{\partial z}\right)\Big|_{\text{at surface}} = \tau^x$$

Surface Stress (e.g. winds)

Dynamic Boundary Conditions

ensuring a dynamic continuity at the boundaries, for a **moving boundary**



<u>Continuity of velocity and **tangential stresses**</u>

$$\tau^x = C_d \, \rho_{\text{air}} \, U_{10} u_{10}, \quad \tau^y = C_d \, \rho_{\text{air}} \, U_{10} v_{10},$$

$$U_{10} = \sqrt{u_{10}^2 + v_{10}^2}$$

Cd= drag coefficient

Wind Speed at 10m

Dynamic Boundary Conditions

ensuring a dynamic continuity at the boundaries, for a **open boundaries**

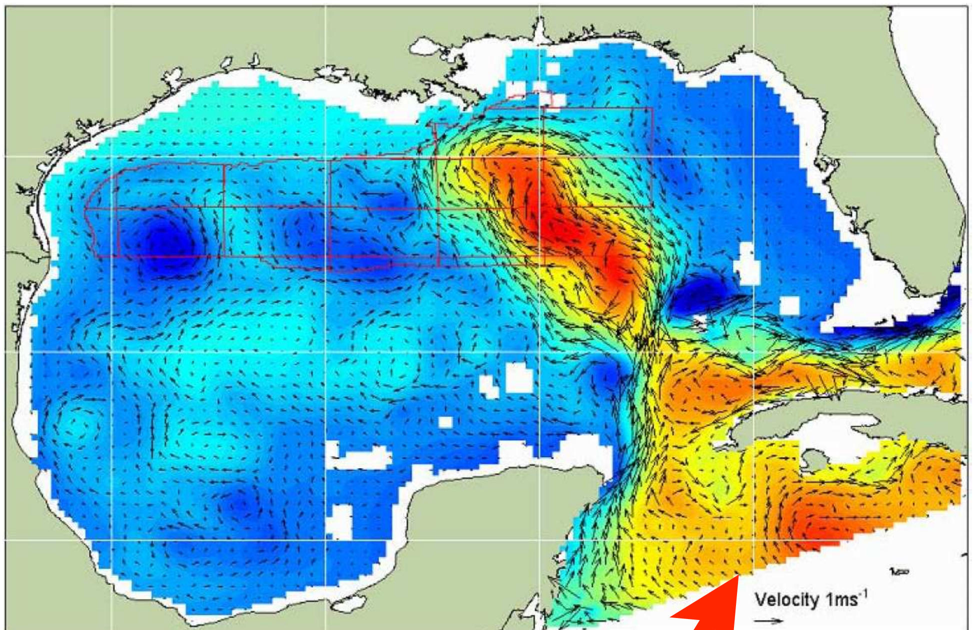


Figure 4-5 Open boundaries are common in regional modeling. Conditions at open boundaries are generally difficult to impose. In particular the nature of the condition depends on whether the flow enters the domain (carrying unknown information from the exterior) or leaves it (exporting known information). (*Courtesy of the HYCOM Consortium on Data-Assimilative Modeling*)

Nesting regional high resolution model raises the issues of prescribing boundary conditions in open ocean regions

Tracer Boundary Conditions

advective and diffusive fluxes of buoyancy (that is fluxes of heat and salt)

$$\text{energy:} \quad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial \rho}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial \rho}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa_E\frac{\partial \rho}{\partial z}\right)$$

Dirichlet condition

prescribing the value of the variable (e.g. advective flux)

Newmann condition

prescribing the values of the gradient (e.g. diffusive flux)

Cauchy condition, Robin condition

prescribing the values of the total flux advective + diffusive

At a solid insulated boundary the total flux is set to zero.

Tracer Boundary Conditions

advective and diffusive fluxes of buoyancy (that is fluxes of heat and salt)

energy:
$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial \rho}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial \rho}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa_E\frac{\partial \rho}{\partial z}\right)$$

Example of surface Boundary Condition

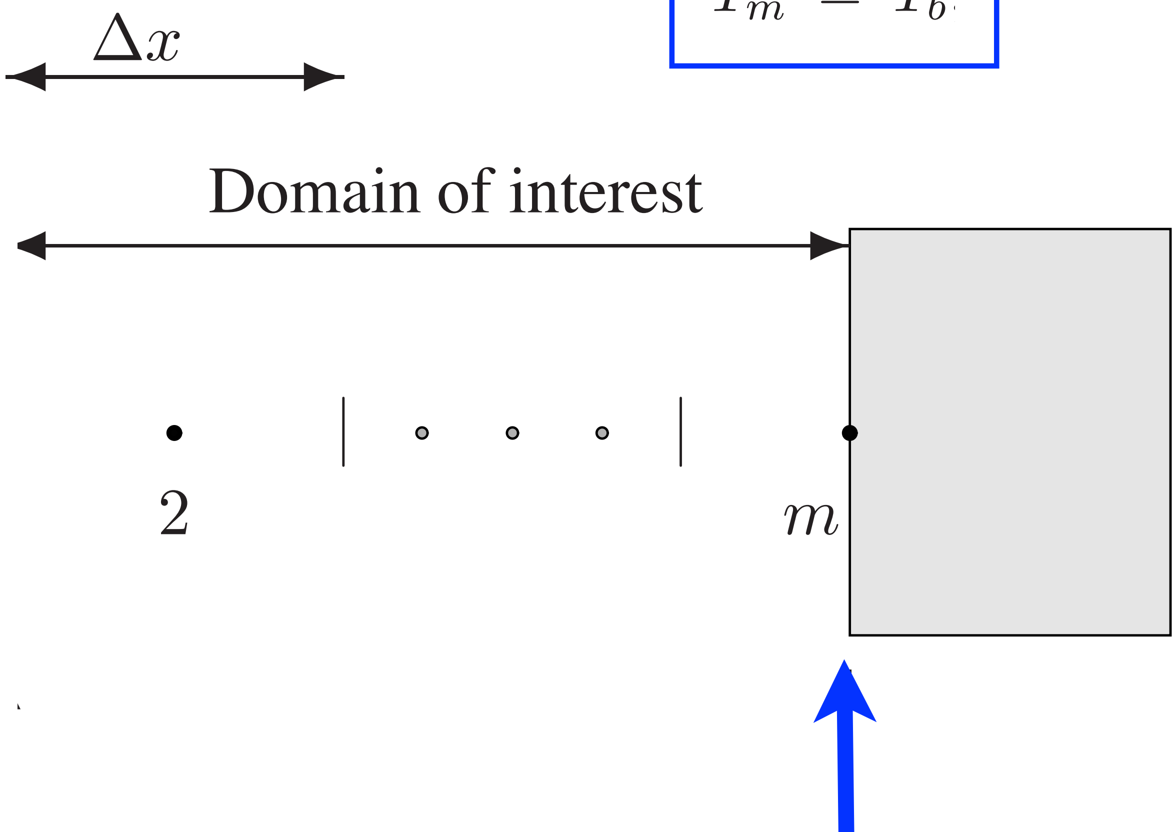
$$-\kappa_T \left.\frac{\partial T}{\partial z}\right|_{z=\eta} = F(T_{\text{sea}}, T_{\text{air}}, \mathbf{u}_{10}, \text{cloudiness}, \text{moisture}, \ldots)$$

Numerical Implementation of Boundary Conditions

Dirichlet condition

prescribing the value of the variable (e.g. advective flux)

$$\tilde{T}_m = T_b$$



Boundary aligned with grid node

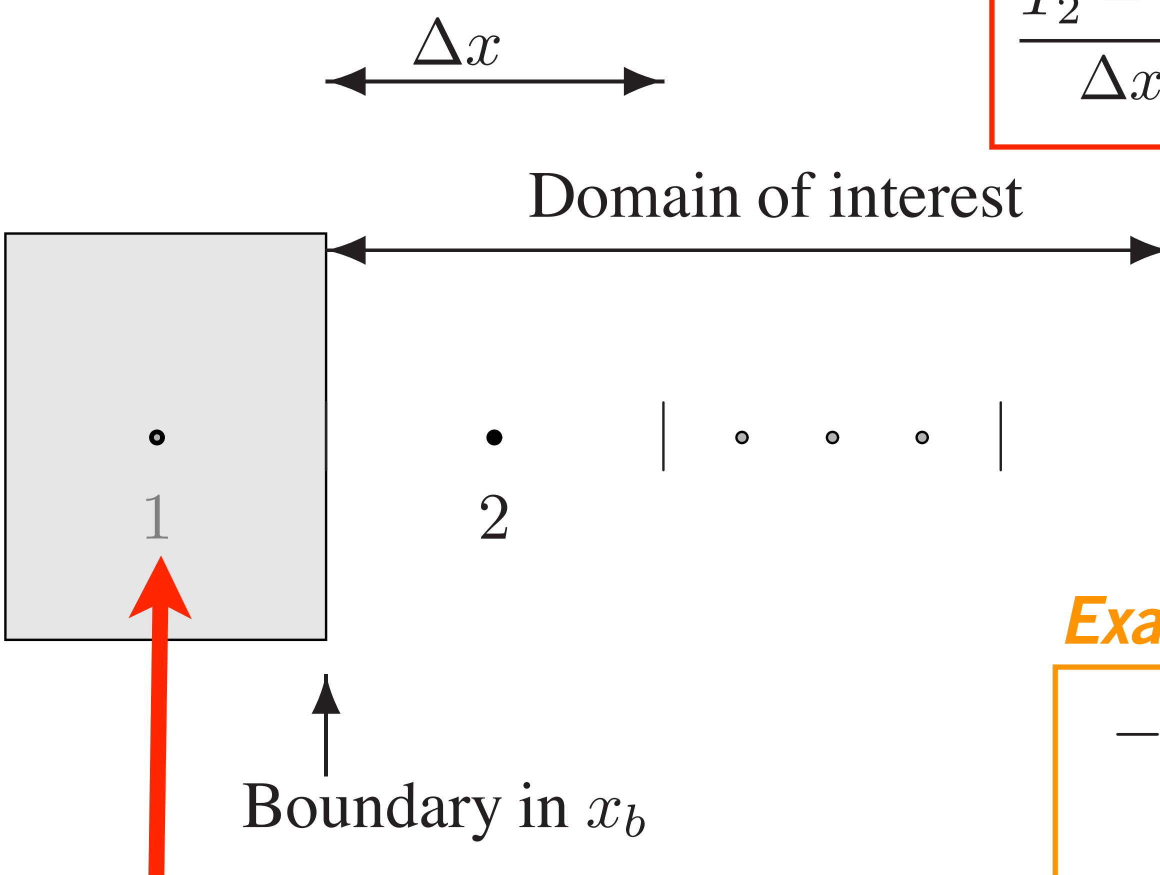
Numerical Implementation of Boundary Conditions

Newmann condition

prescribing the values of the gradient (e.g. diffusive flux)

$$\frac{\tilde{T}_2 - \tilde{T}_1}{\Delta x} \simeq \left.\frac{\partial T}{\partial x}\right|_{x_b}$$

centered scheme, 2nd order



Additional nodes (or grid points) facilitate the implementation of boundary conditions

Example

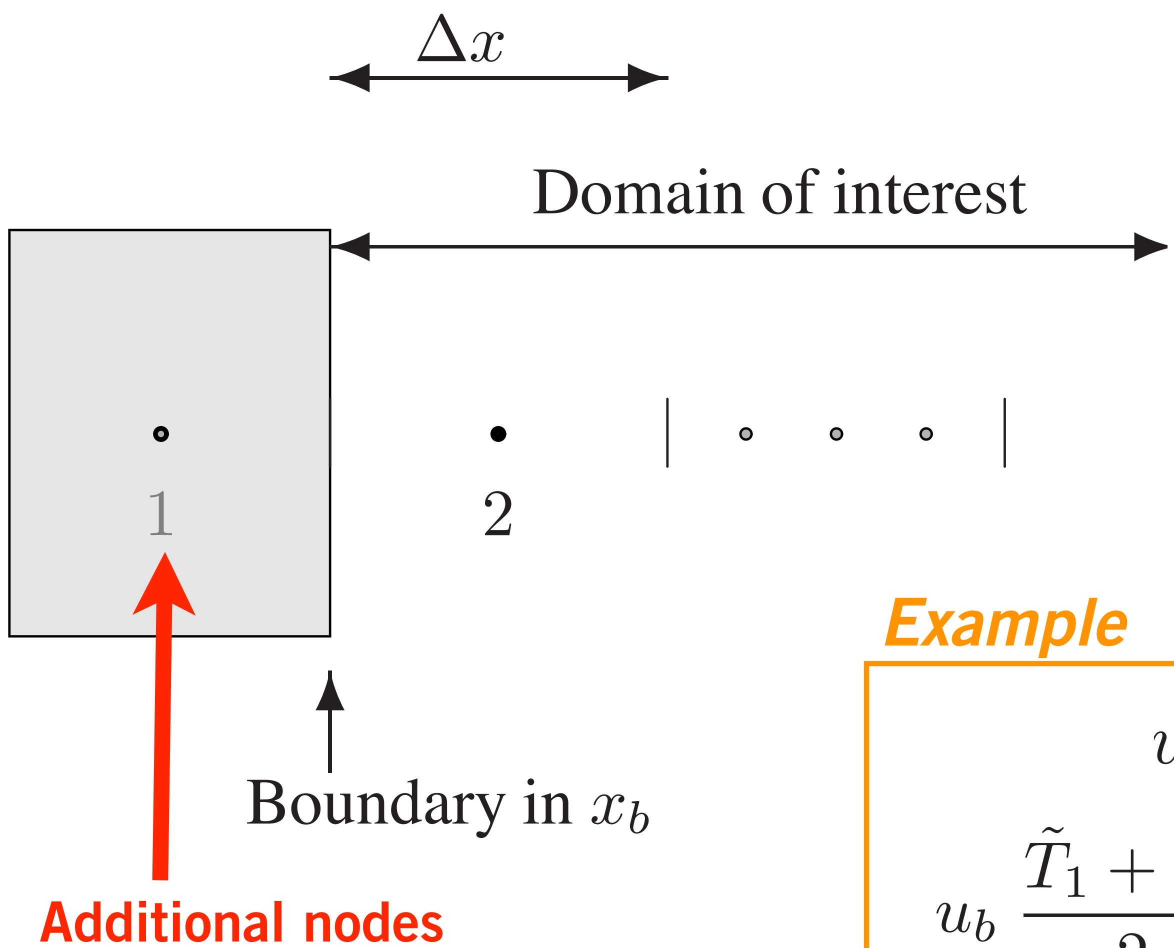
$$-\kappa_T(\partial T/\partial x) = q_b$$

$$\tilde{T}_1 = \tilde{T}_2 + \Delta x \frac{q_b}{\kappa_T}$$

Numerical Implementation of Boundary Conditions

Cauchy condition, Robin condition

prescribing the values of the total flux advective + diffusive



Example

$$uT - \kappa_T(\partial T/\partial x) = q_b$$

$$u_b \frac{\tilde{T}_1 + \tilde{T}_2}{2} - \kappa_T \frac{\tilde{T}_2 - \tilde{T}_1}{\Delta x} = q_b$$

advective — diffusive

Numerical Implementation of Boundary Conditions

higher order operators ..

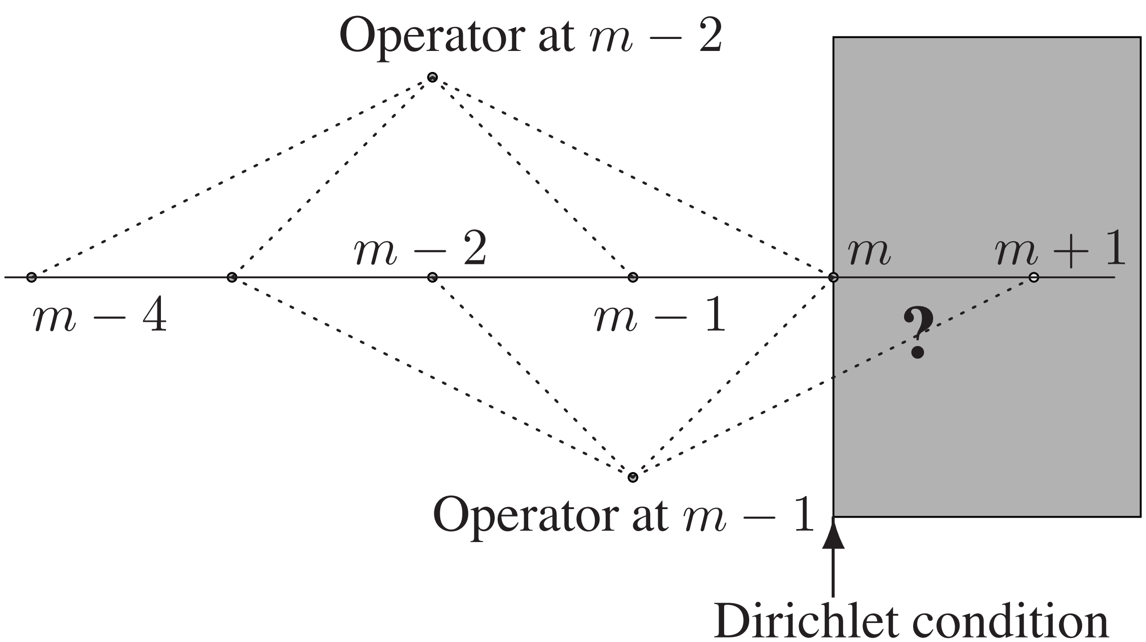


Figure 4-7 An operator spanning 2 points on each side of the calculation point can be applied only up to $m-2$ if a single Dirichlet condition is prescribed. When applying the same operator at $m-1$ we face the problem that the value at $m+1$ does not exist.

Accuracy and Errors

- *Modeling errors*: This error is caused by the imperfections of the mathematical model in representing the physical system. It is thus the difference between the evolution of the real system and that of the exact solution of its mathematical representation. Earlier in this chapter we introduced simplifications to the equations and added parameterizations of unresolved processes, which all introduce errors of representation. Furthermore, even if the model formulation had been ideal, coefficients remain imperfecly known. Uncertainties in the accompanying boundary conditions also contribute to modeling errors.

- *Discretization errors*: This error is introduced when the original equations are approximated to transform them into a computer code. It is thus the difference between the exact solution of the continuous problem and the exact numerical solution of the discretized equations. Examples are the replacement of derivatives by finite differences and the use of guesses in predictor-corrector schemes.

- *Iteration errors*: This error originates with the use of iterative methods to perform intermediate steps in the algorithm and is thus measured as the difference between the exact solution of the discrete equations and the numerical solution actually obtained. An example is the use of the so-called Jacobi method to invert a matrix at some stage of the calculations: for the sake of time, the iterative process is interrupted before full convergence is reached.

- *Rounding errors*: These errors are due to the fact that only a finite number of digits are used in the computer to represent real numbers.

A well constructed model should ensure that

rounding errors $\ll$ iteration errors $\ll$ discretization errors $\ll$ modeling errors.

Surface and Bottom Boundary Conditions for Momentum, the Ekman boundary Layers

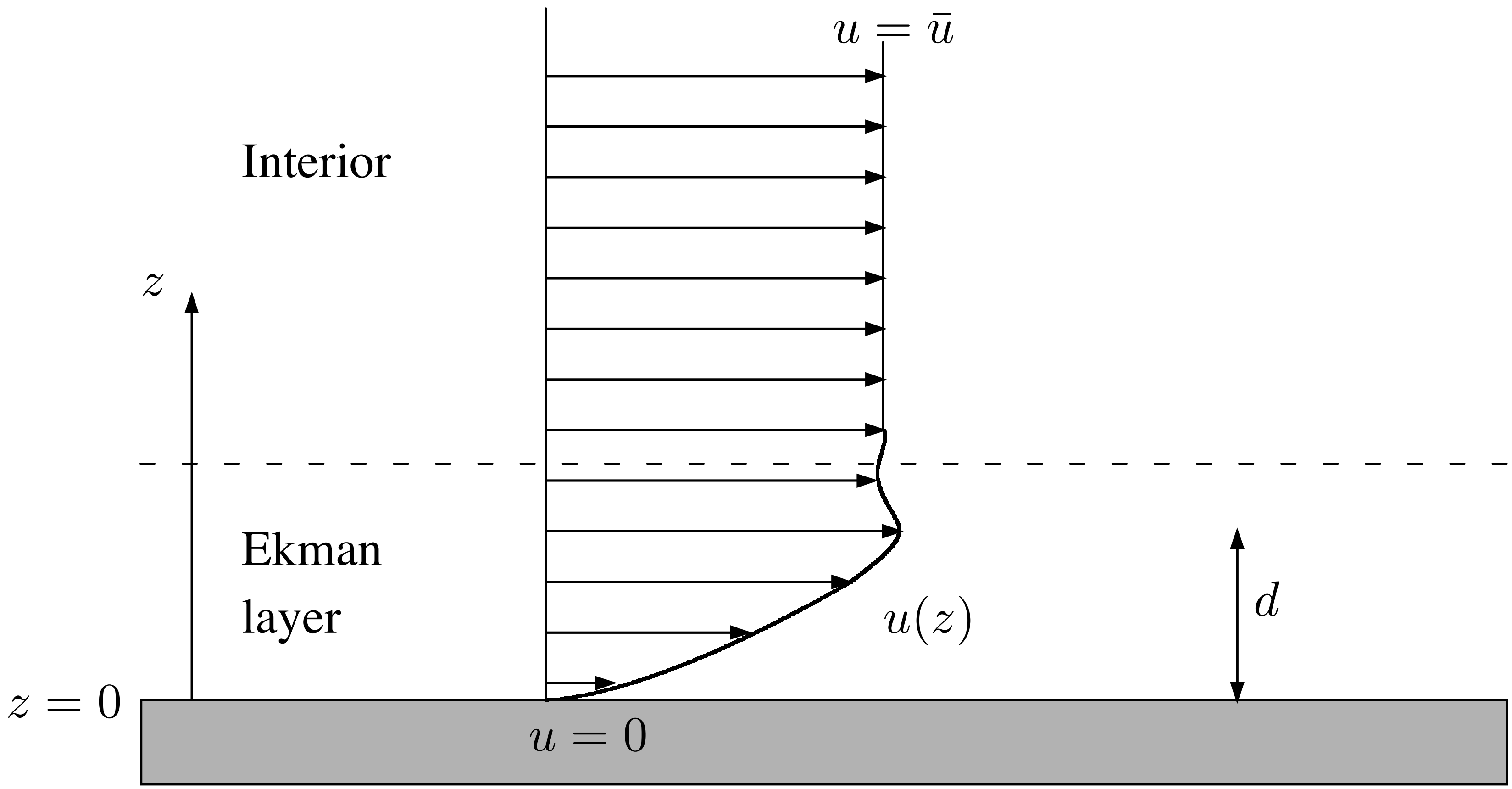


Figure 8-3 Frictional influence of a flat bottom on a uniform flow in a rotating framework.

Surface and Bottom Boundary Conditions for Momentum, the Ekman boundary Layers

$x-\text{momentum:}$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} - fv =$$

$$-\frac{1}{\rho_0}\,\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial u}{\partial y}\right) + \boxed{\frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right)}$$

surface and bottom boundary conditions for vertical derivative

Surface and Bottom Boundary Conditions for Momentum, the Ekman boundary Layers

$$x-\text{momentum:}\quad \frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}+w\,\frac{\partial u}{\partial z}-fv=$$

$$-\frac{1}{\rho_0}\,\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial u}{\partial y}\right)+\boxed{\frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right)}$$

surface and bottom boundary conditions for vertical derivative

inertial terms | *rotation* | *frictional forces*

$$\boxed{\frac{1}{\Omega T},\quad \frac{U}{\Omega L},\quad \frac{U}{\Omega L},\quad \frac{WL}{UH}\cdot\frac{U}{\Omega L}},\ \boxed{1,}\quad \frac{P}{\rho_0\Omega LU},\ \boxed{\frac{\mathcal{A}}{\Omega L^2},\quad \frac{\nu_E}{\Omega H^2}}$$

Surface and Bottom Boundary Conditions for Momentum, the Ekman boundary Layers

Ekman Number

viscous force/rotation

$$Ek = \frac{\nu_E}{\Omega H^2}$$

inertial terms *rotation* *frictional forces*

$$\frac{1}{\Omega T}, \quad \frac{U}{\Omega L}, \quad \frac{U}{\Omega L}, \quad \frac{WL}{UH} \cdot \frac{U}{\Omega L}, \quad 1, \quad \frac{P}{\rho_0 \Omega L U}, \quad \frac{\mathcal{A}}{\Omega L^2}, \quad \frac{\nu_E}{\Omega H^2}$$

Surface and Bottom Boundary Conditions for Momentum, the Ekman boundary Layers

Ekman Number

viscous force/rotation

$$Ek = \frac{\nu_E}{\Omega H^2}$$

Ekman number typically small and friction is neglected, except close to the boundary ~1

inertial terms *rotation* *frictional forces*

$$\frac{1}{\Omega T}, \quad \frac{U}{\Omega L}, \quad \frac{U}{\Omega L}, \quad \frac{WL}{UH} \cdot \frac{U}{\Omega L}, \quad 1, \quad \frac{P}{\rho_0 \Omega L U}, \quad \frac{\mathcal{A}}{\Omega L^2}, \quad \frac{\nu_E}{\Omega H^2}$$

Surface and Bottom Boundary Conditions for Momentum, the Ekman boundary Layers

Ekman Number

viscous force/rotation

$$Ek = \frac{\nu_E}{\Omega H^2}$$

Ekman number typically small and friction is neglected, except close to the boundary ~1

1

$$d \sim \sqrt{\frac{\nu_E}{\Omega}} \, .$$

with d << H

inertial terms | *rotation* | *frictional forces*

$$\frac{1}{\Omega T}, \quad \frac{U}{\Omega L}, \quad \frac{U}{\Omega L}, \quad \frac{WL}{UH} \cdot \frac{U}{\Omega L}, \quad 1, \quad \frac{P}{\rho_0 \Omega L U}, \quad \frac{\mathcal{A}}{\Omega L^2}, \quad \frac{\nu_E}{\Omega H^2}$$

Surface and Bottom Boundary Conditions for Momentum, the Ekman boundary Layers

Dominant balance is between ***rotation*** and ***friction terms***

rotation ***frictional forces***

$$, \boxed{1}, \quad \frac{P}{\rho_0 \Omega L U}, \quad \boxed{\frac{\mathcal{A}}{\Omega L^2}, \quad \frac{\nu_E}{\Omega H^2}}$$

Surface and Bottom Boundary Conditions for Momentum, the Ekman boundary Layers

1 3

$$x-\text{momentum:} \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \boxed{- fv =}$$

$$\boxed{-\frac{1}{\rho_0}\frac{\partial p}{\partial x}} + \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial u}{\partial y}\right) + \boxed{\frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right)}$$

2

Dominant balance is between *rotation* and *vertical friction terms*

rotation *frictional forces*

$$, \boxed{1,} \quad \frac{P}{\rho_0 \Omega L U}, \quad \boxed{\frac{\mathcal{A}}{\Omega L^2}, \quad \frac{\nu_E}{\Omega H^2}}$$

Surface and Bottom Boundary Conditions for Momentum, the Ekman boundary Layers

$$
\begin{aligned}
- \; f\,(v - \bar{v}) \;&=\; \nu_E \,\frac{\partial^2 u}{\partial z^2} \\
+ \, f\,(u - \bar{u}) \;&=\; \nu_E \,\frac{\partial^2 v}{\partial z^2} \\
\text{Surface } (z \;=\; 0): \qquad & \rho_0 \nu_E \frac{\partial u}{\partial z} \;=\; \tau^x, \;\; \rho_0 \nu_E \frac{\partial v}{\partial z} \;=\; \tau^y \\
\text{Toward interior } (z \;\rightarrow\; -\infty): \qquad & u \;=\; \bar{u}, \quad v \;=\; \bar{v}.
\end{aligned}
$$

Surface and Bottom Boundary Conditions for Momentum, the Ekman surface Layers

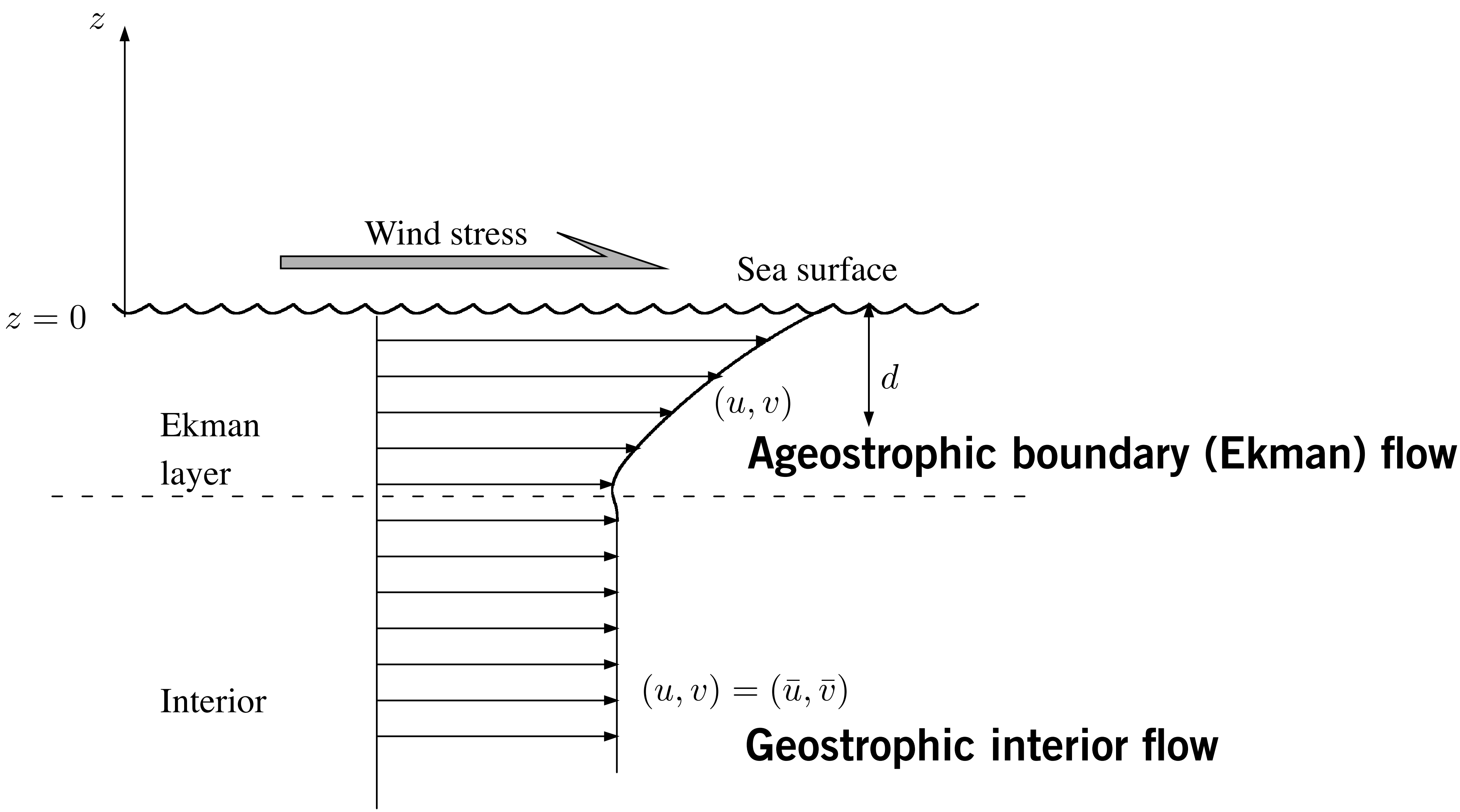


Figure 8-6 The surface Ekman layer generated by a wind stress on the ocean.

Surface and Bottom Boundary Conditions for Momentum, the Ekman surface Layers

$$- f\,(v - \bar{v}) = \nu_E \frac{\partial^2 u}{\partial z^2}$$

$$+ f\,(u - \bar{u}) = \nu_E \frac{\partial^2 v}{\partial z^2}$$

$$\text{Surface } (z = 0): \quad \rho_0 \nu_E \frac{\partial u}{\partial z} = \tau^x, \quad \rho_0 \nu_E \frac{\partial v}{\partial z} = \tau^y$$

$$\text{Toward interior } (z \to -\infty): \quad u = \bar{u}, \quad v = \bar{v}.$$

Surface and Bottom Boundary Conditions for Momentum, the Ekman surface Layers

$$-\ f\ (v - \bar{v}) \ = \ \nu_E \frac{\partial^2 u}{\partial z^2}$$

$$+\ f\ (u - \bar{u}) \ = \ \nu_E \frac{\partial^2 v}{\partial z^2}$$

$$\text{Surface } (z \ = \ 0): \qquad \rho_0 \nu_E \frac{\partial u}{\partial z} \ = \ \tau^x, \ \ \rho_0 \nu_E \frac{\partial v}{\partial z} \ = \ \tau^y$$

$$\text{Toward interior } (z \ \rightarrow \ -\infty): \qquad u \ = \ \bar{u}, \ \ v \ = \ \bar{v}.$$

Ekman Transport

$$\mathrm{U} \ = \ \int_{-\infty}^{0} (u - \bar{u})\, dz \ = \ \frac{1}{\rho_0 f}\, \tau^y$$

$$\mathrm{V} \ = \ \int_{-\infty}^{0} (v - \bar{v})\, dz \ = \ \frac{-1}{\rho_0 f}\, \tau^x.$$

Surface and Bottom Boundary Conditions for Momentum, the Ekman surface Layers

$$-\ f\ (v - \bar{v}) \;=\; \nu_E\ \frac{\partial^2 u}{\partial z^2}$$

$$+\ f\ (u - \bar{u}) \;=\; \nu_E\ \frac{\partial^2 v}{\partial z^2}$$

$$\text{Surface}\ (z \;=\; 0)\ : \qquad \rho_0 \nu_E \frac{\partial u}{\partial z} \;=\; \tau^x, \quad \rho_0 \nu_E \frac{\partial v}{\partial z} \;=\; \tau^y$$

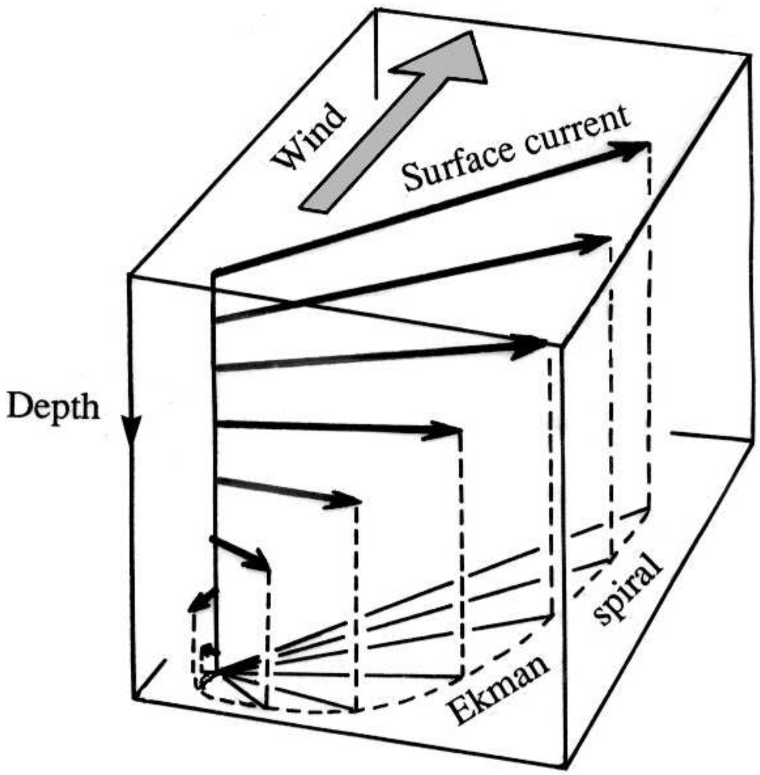
$$\text{Toward interior}\ (z \;\rightarrow\; -\infty)\ : \qquad u \;=\; \bar{u}, \quad v \;=\; \bar{v}.$$

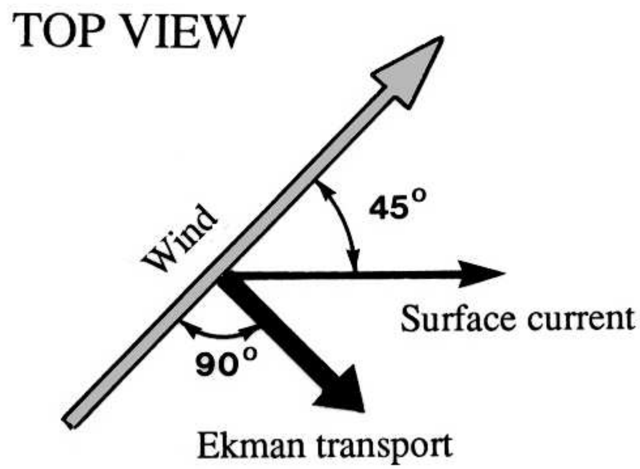
<u>Solution</u>

$$u \;=\; \bar{u} + \frac{\sqrt{2}}{\rho_0 f d}\ e^{z/d} \left[\tau^x\ \cos\left(\frac{z}{d} - \frac{\pi}{4}\right) - \tau^y\ \sin\left(\frac{z}{d} - \frac{\pi}{4}\right)\right]$$

$$v \;=\; \bar{v} + \frac{\sqrt{2}}{\rho_0 f d}\ e^{z/d} \left[\tau^x\ \sin\left(\frac{z}{d} - \frac{\pi}{4}\right) + \tau^y\ \cos\left(\frac{z}{d} - \frac{\pi}{4}\right)\right]$$

Surface and Bottom Boundary Conditions for Momentum, the Ekman surface Layers





Solution

$$u = \bar{u} + \frac{\sqrt{2}}{\rho_0 f d} e^{z/d} \left[\tau^x \cos\left(\frac{z}{d} - \frac{\pi}{4}\right) - \tau^y \sin\left(\frac{z}{d} - \frac{\pi}{4}\right)\right]$$

$$v = \bar{v} + \frac{\sqrt{2}}{\rho_0 f d} e^{z/d} \left[\tau^x \sin\left(\frac{z}{d} - \frac{\pi}{4}\right) + \tau^y \cos\left(\frac{z}{d} - \frac{\pi}{4}\right)\right]$$

Implication of the Ekman surface Layers

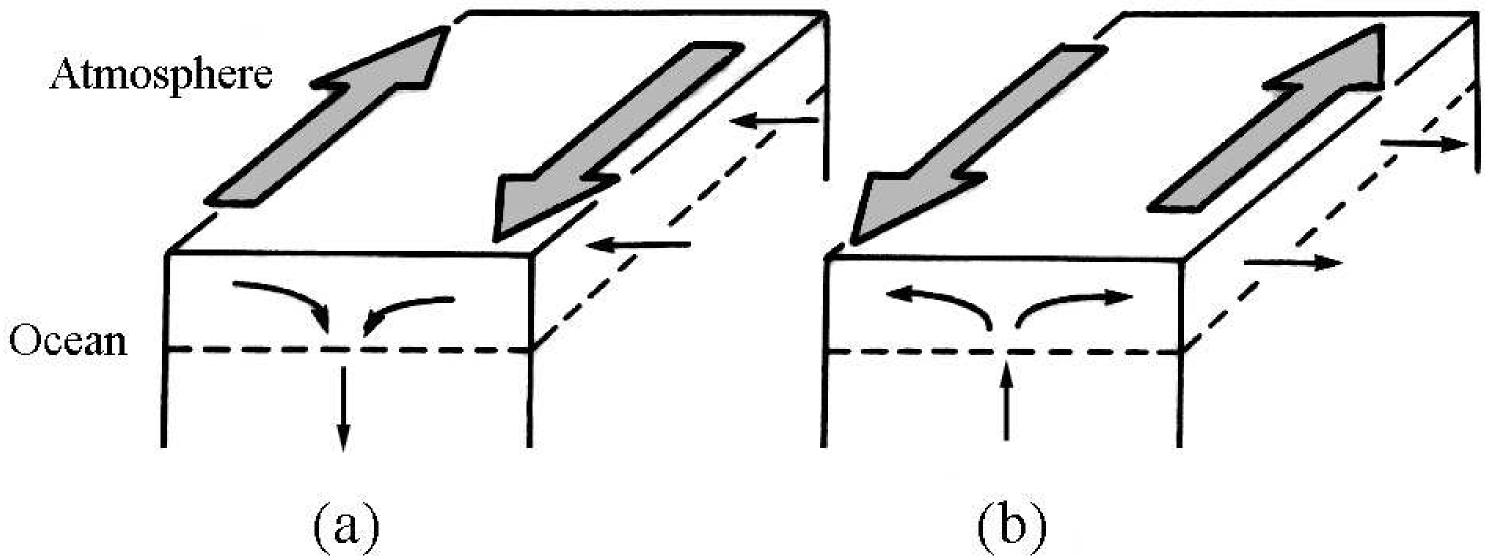


Figure 8-8 Ekman pumping in an ocean subject to sheared winds (case of Northern Hemisphere).

Surface and Bottom Boundary Conditions for Momentum, the Ekman bottom Layers

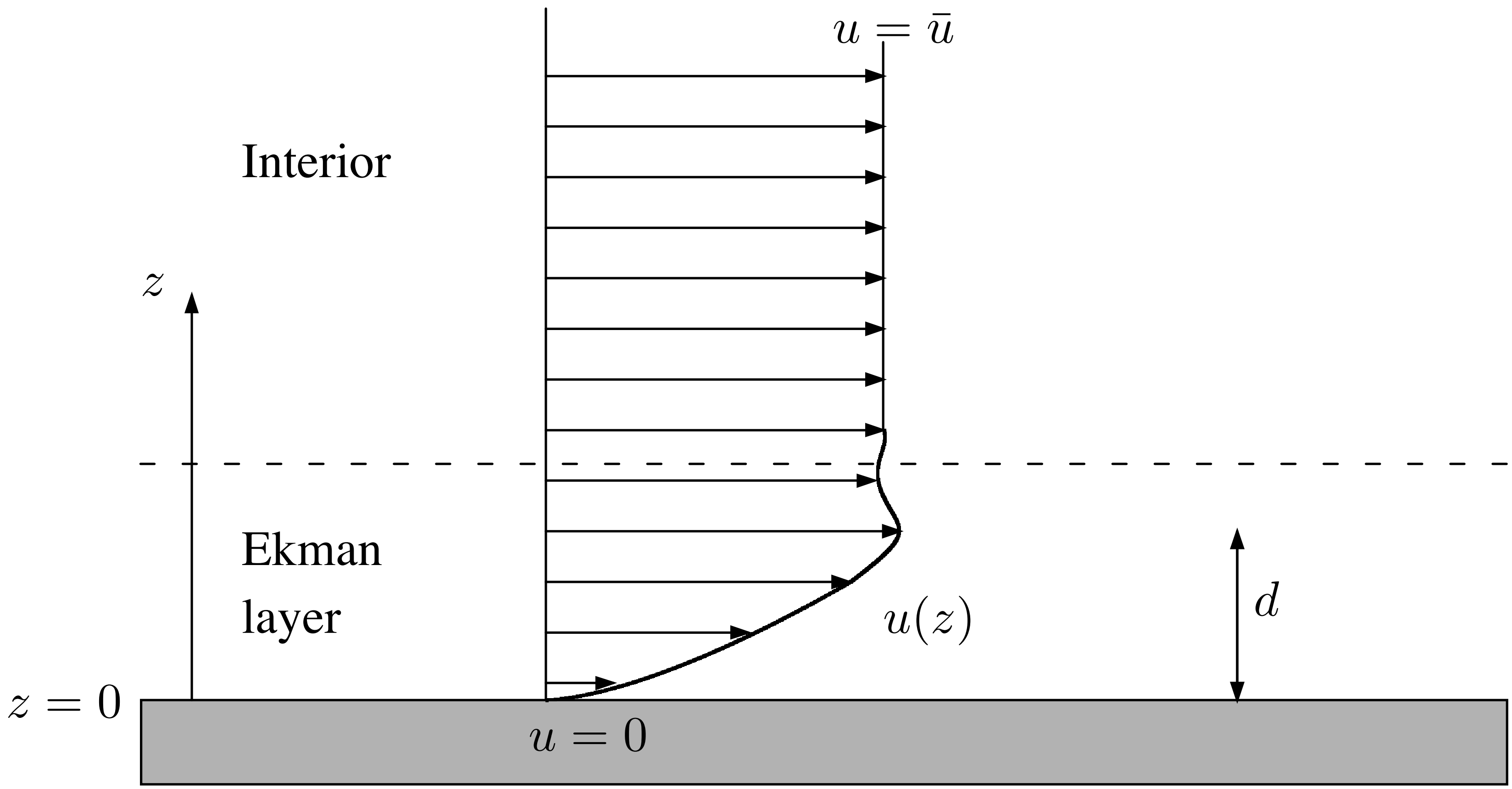


Figure 8-3 Frictional influence of a flat bottom on a uniform flow in a rotating framework.

Surface and Bottom Boundary Conditions for Momentum, the Ekman bottom Layers

$$
\begin{aligned}
-\ fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu_E \frac{\partial^2 u}{\partial z^2} \\
+\ fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu_E \frac{\partial^2 v}{\partial z^2} \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z},
\end{aligned}
$$

$$
\begin{aligned}
\text{Bottom } (z = 0): &\quad u = 0, \quad v = 0, \\
\text{Toward the interior } (z \gg d): &\quad u = \bar{u}, \quad v = 0, \quad p = \bar{p}(x, y).
\end{aligned}
$$

Surface and Bottom Boundary Conditions for Momentum, the Ekman bottom Layers

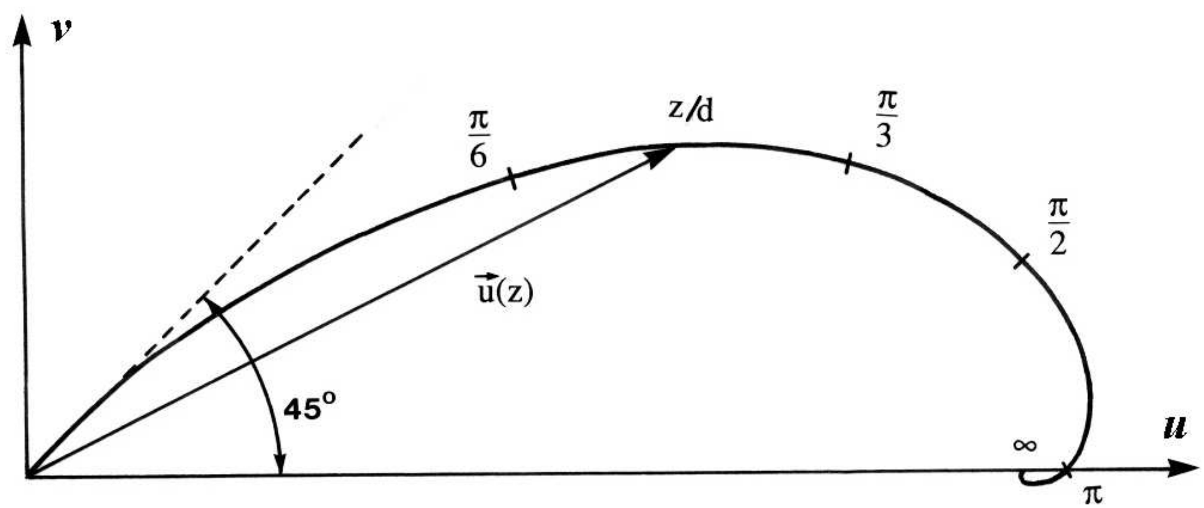


Figure 8-4 The velocity spiral in the bottom Ekman layer. The figure is drawn for the Northern Hemisphere ($f > 0$), and the deflection is to the left of the current above the layer. The reverse holds for the Southern Hemisphere.