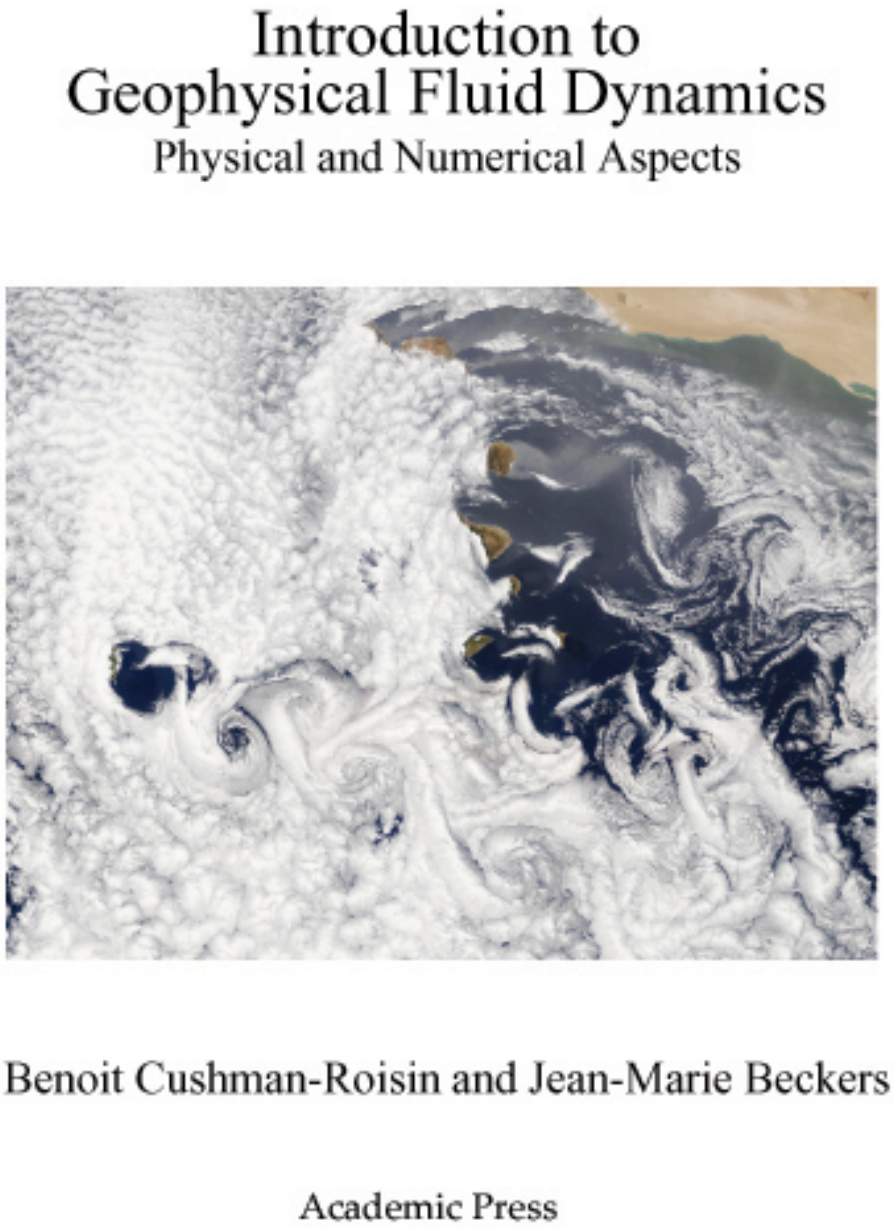
Ocean Modeling - EAS 8803

Linear Waves

- Linear equations for unsteady flows
- Waves along boundaries - Kelvin Waves
- Inertial-gravity waves
- Planetary waves - Rossby Waves
- Topographic Waves


Introduction to
Geophysical Fluid Dynamics
Physical and Numerical Aspects

Benoit Cushman-Roisin and Jean-Marie Beckers

Academic Press

Chapter 9

Equations of Geostrophic homogeneous flows

shallow-water model or *barotropic equations*

describe unsteady motions of a 2D uniform density layer

or

of the depth average motion

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} - fv &= -\,g\,\frac{\partial \eta}{\partial x}\\
\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + fu &= -\,g\,\frac{\partial \eta}{\partial y}
\end{aligned}
$$

$$
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0
$$

Linear Equations and waves solutions for homogeneous flows

$$\boxed{Ro = \frac{U}{\Omega L} \ll 1.}$$

$$\boxed{Ro_T = \frac{1}{\Omega T} \sim 1}$$

weak/large-scale flows that evolve relatively fast **= Waves**

$$\frac{\partial u}{\partial t} + u \cancel{\frac{\partial u}{\partial x}}_{\text{small}} + v \cancel{\frac{\partial u}{\partial y}}_{\text{small}} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + u \cancel{\frac{\partial v}{\partial x}}_{\text{small}} + v \cancel{\frac{\partial v}{\partial y}}_{\text{small}} + fu = -g\frac{\partial \eta}{\partial y}$$

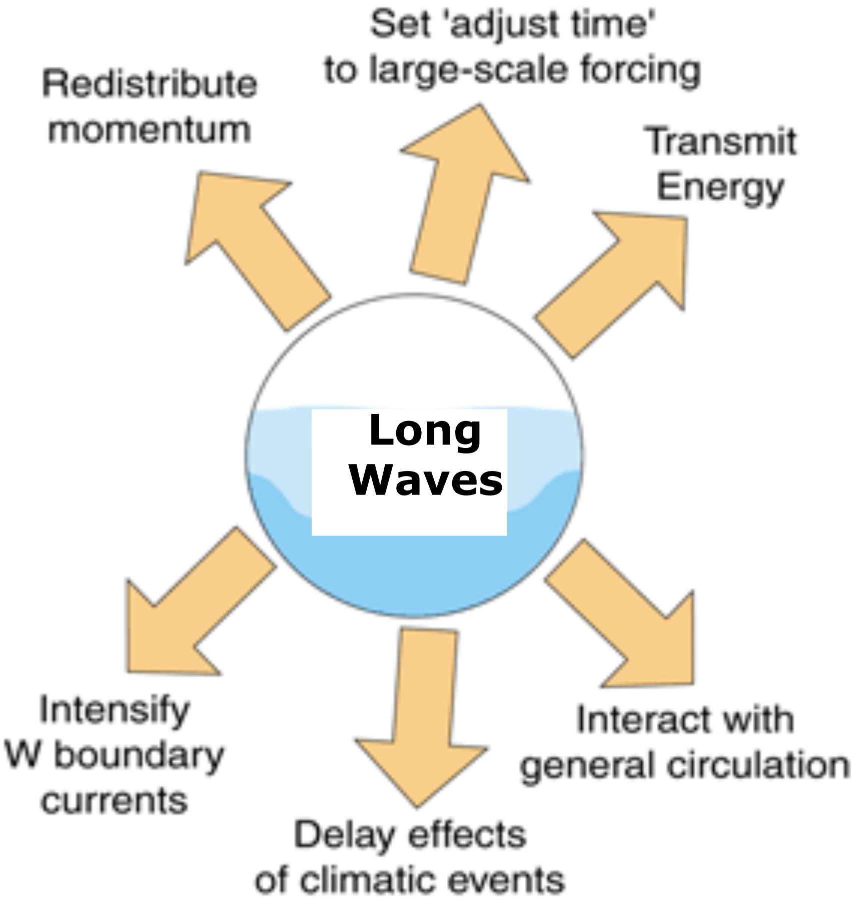
$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0,$$

Linear Equations and waves solutions for homogeneous flows

$$\boxed{Ro = \frac{U}{\Omega L} \ll 1.}$$

$$\boxed{Ro_T = \frac{1}{\Omega T} \sim 1}$$

weak/large-scale flows that evolve relatively fast



Linear Equations and waves solutions for homogeneous flows

$$\boxed{Ro = \frac{U}{\Omega L} \ll 1.}$$

$$\boxed{Ro_T = \frac{1}{\Omega T} \sim 1}$$

weak/large-scale flows that evolve relatively fast

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0,$$

recall

$$\boxed{\eta = h - H,}$$

Linear Equations and waves solutions for homogeneous flows

1

$$\boxed{Ro = \frac{U}{\Omega L} \ll 1.}$$

$$\boxed{Ro_T = \frac{1}{\Omega T} \sim 1}$$

2

weak/large-scale flows that evolve relatively fast

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

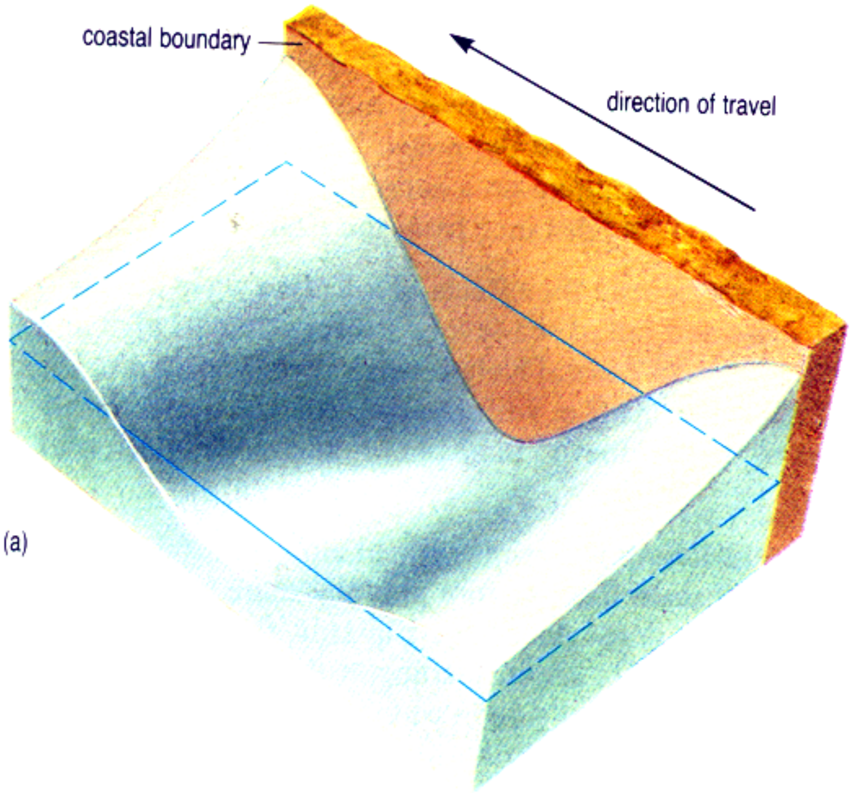
small amplitude waves

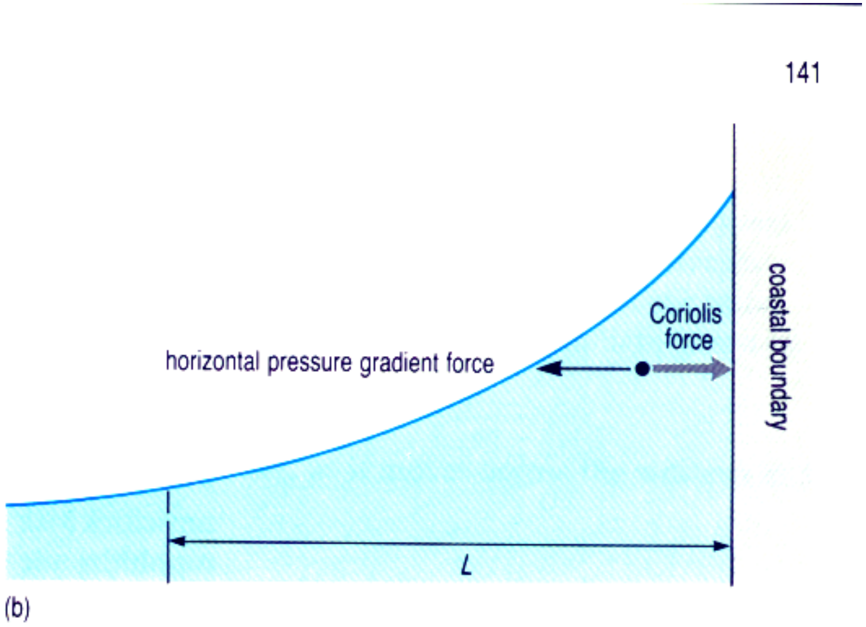
$$\boxed{\Delta H \ll H}$$

3

Kelvin Waves

waves moving along a side boundary with Ro << 1





$$R = \frac{\sqrt{gH}}{f} = \frac{c}{f}$$

Rossby radius of deformation

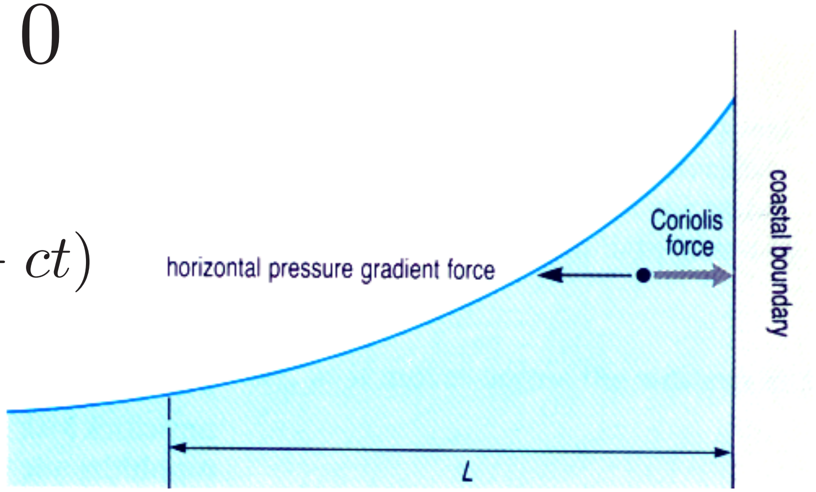
Kelvin Waves

waves moving along a side boundary with Ro << 1

Boundary Conditions: $u = 0$

General Solution:

$$v = V_1(x,\, y+ct) + V_2(x,\, y-ct)$$



$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial y^2}$$

$$c = \sqrt{gH}$$

phase speed of wave does not depend on wave number = non dispersive

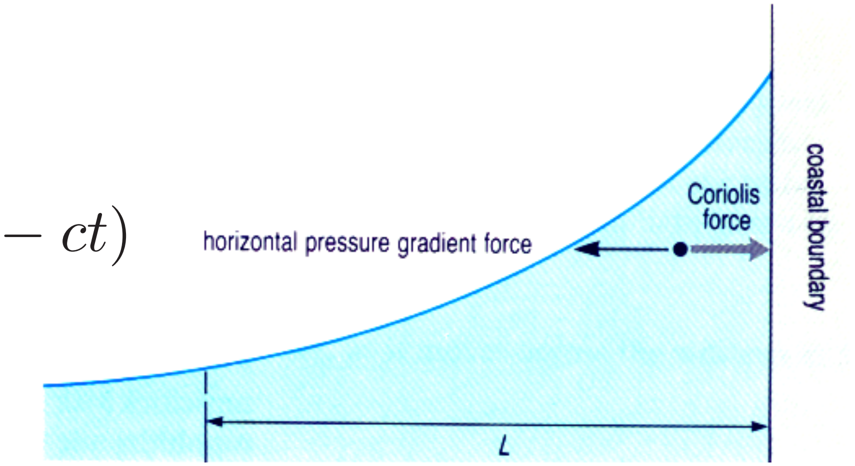
Governing Equations:

vanish

$$\left\{ \begin{aligned} \cancel{\frac{\partial u}{\partial t}} - fv &= -g\frac{\partial \eta}{\partial x} \\ \frac{\partial v}{\partial t} + \cancel{fu} &= -g\frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial t} + H\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y}\right) &= 0 \end{aligned} \right.$$

Kelvin Waves

waves moving along a side boundary with Ro << 1



General Solution:

$$v \;=\; V_1(x,\; y + ct) \;+\; V_2(x,\; y - ct)$$

$$\eta \;=\; -\sqrt{\frac{H}{g}}\, V_1(x,\; y + ct) \;+\; + \sqrt{\frac{H}{g}}\, V_2(x,\; y - ct)$$

$$\frac{\partial V_1}{\partial x} \;=\; -\,\frac{f}{\sqrt{gH}}\, V_1$$

$$\frac{\partial V_2}{\partial x} \;=\; +\,\frac{f}{\sqrt{gH}}\, V_2$$

Governing Equations:

$$-\,fv \;=\; -\,g\,\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} \;=\; -\,g\,\frac{\partial \eta}{\partial y}$$

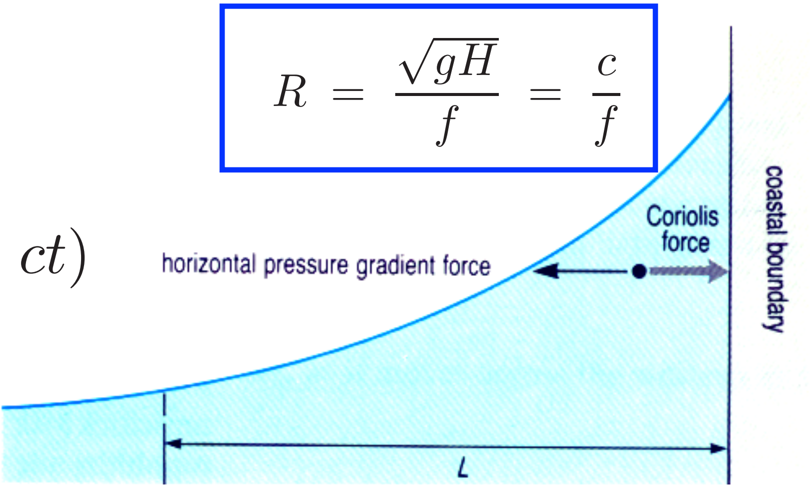
$$\frac{\partial \eta}{\partial t} + H\left(\qquad \frac{\partial v}{\partial y}\right) \;=\; 0$$

Kelvin Waves

waves moving along a side boundary with Ro << 1

Rossby Deformation Radius

$$\boxed{R = \frac{\sqrt{gH}}{f} = \frac{c}{f}}$$



General Solution:

$$v = V_1(x,\, y + ct) + V_2(x,\, y - ct)$$

$$\eta = -\sqrt{\frac{H}{g}}\, V_1(x,\, y+ct) + \\ + \sqrt{\frac{H}{g}}\, V_2(x,\, y-ct)$$

$$V_1 = V_{10}(y + ct)\, e^{-x/\boxed{R}}$$

$$V_2 = V_{20}(y - ct)\, e^{+x/\boxed{R}}$$

Governing Equations:

$$-fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y}$$

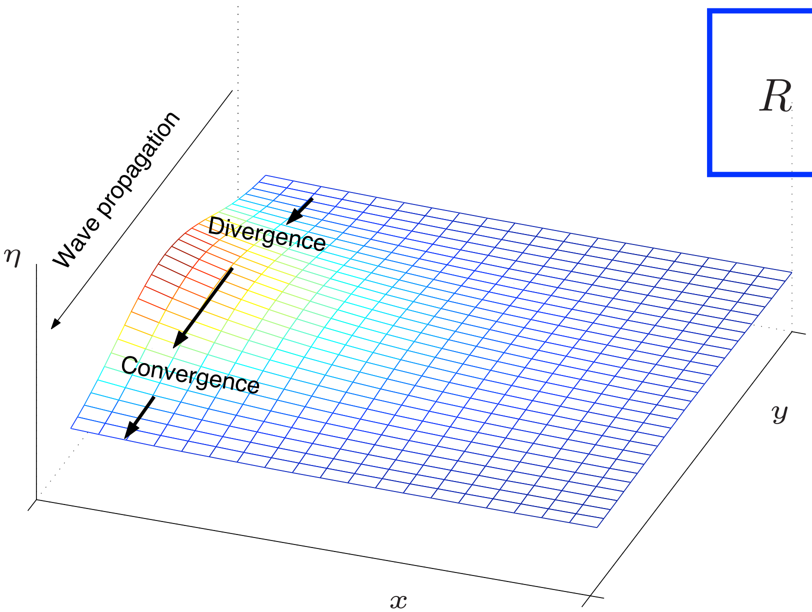
$$\frac{\partial \eta}{\partial t} + H\left(\qquad \frac{\partial v}{\partial y}\right) = 0$$

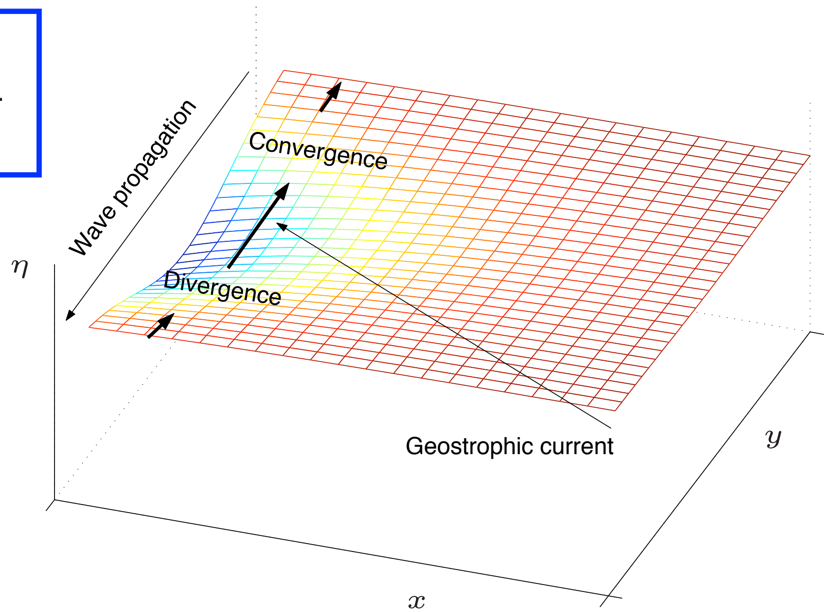
Kelvin Waves

waves moving along a side boundary with Ro << 1

Rossby Deformation Radius

$$\boxed{R = \frac{\sqrt{gH}}{f} = \frac{c}{f}}$$



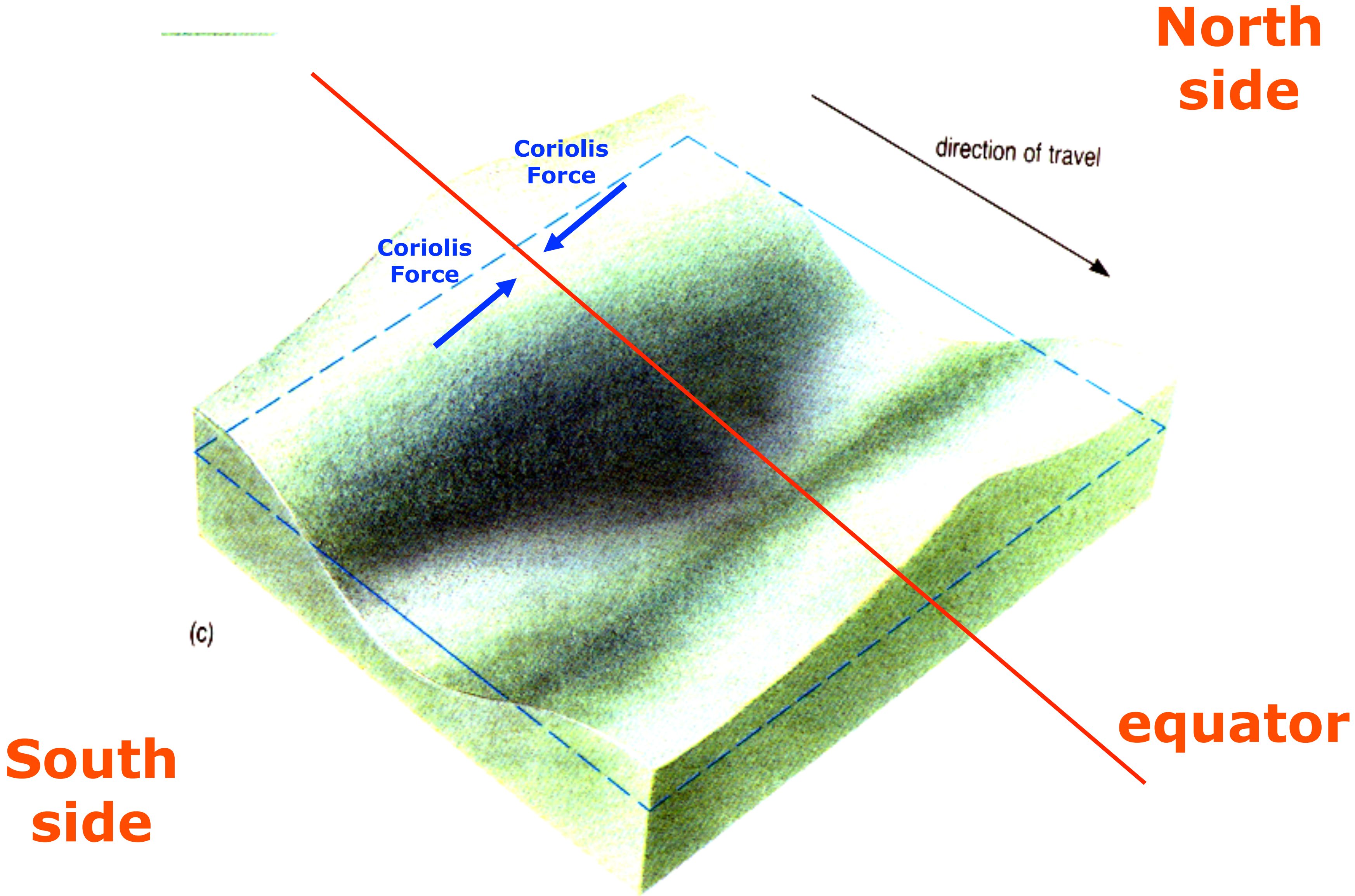


General Solution:

$$\begin{aligned} u &= 0 \\ v &= \sqrt{gH}\ F(y+ct)\, e^{-x/R} \\ \eta &= -\,H\ F(y+ct)\, e^{-x/R}, \end{aligned}$$

Equatorial Kelvin Waves

waves moving along a side boundary (the equator) with Ro << 1



BAROTROPIC

Rossby Deformation Radius

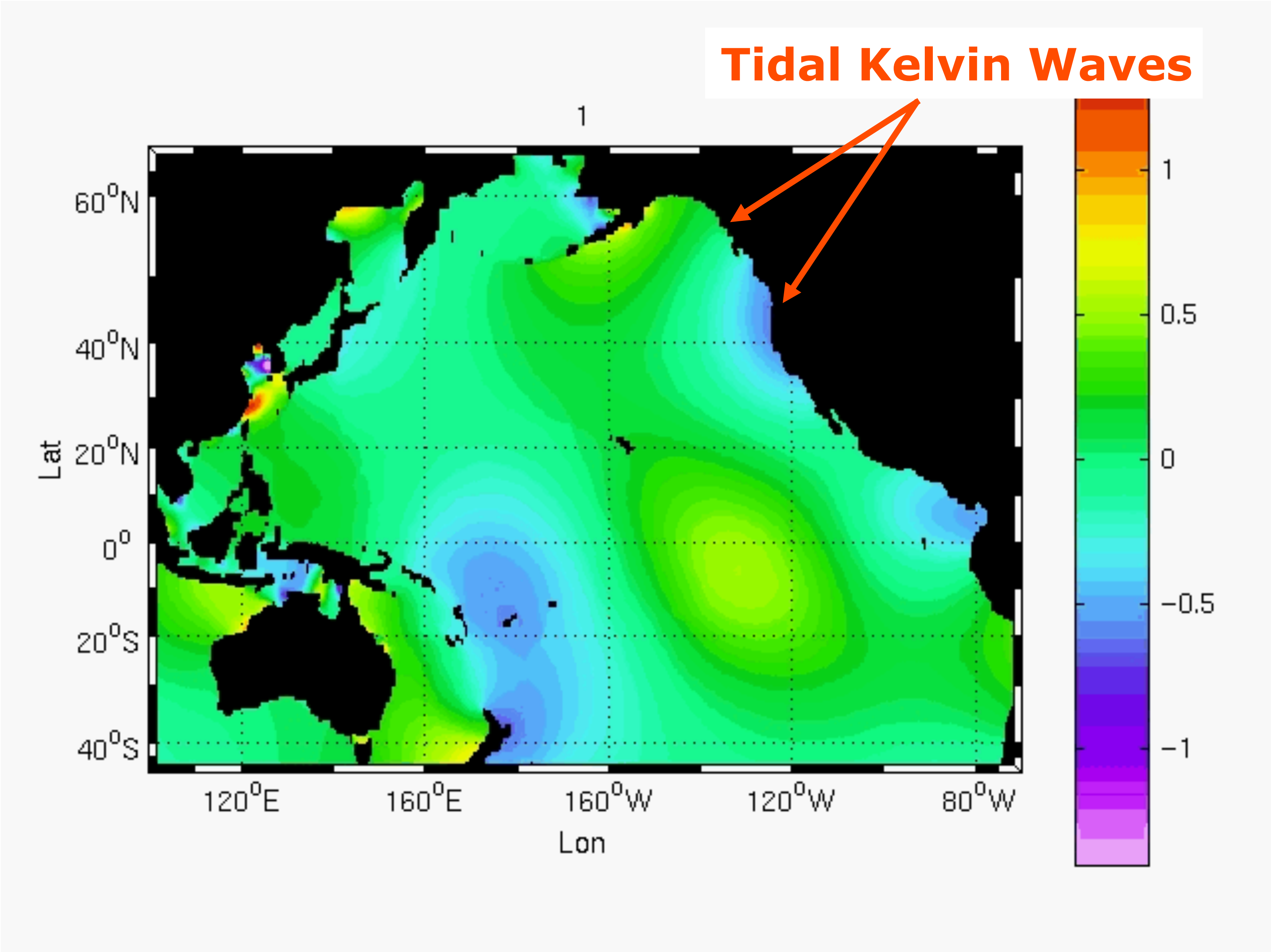
$$\boxed{R = \frac{\sqrt{gH}}{f} = \frac{c}{f}}$$

Definition:

Distance that a particle or wave moving at a certain speed needs to cover in order to be affected by the rotation of the planet.

If *d* is the total depth of the water, we call this the BAROTROPIC DEFORMATION RADIUS or EXTERNAL RADIUS

Animation of Tidal Elevations in the Pacific



Tidal Kelvin Waves

Surface Wave

Wave at the interface between ocean (considered as a water mass of same density) and the atmosphere

Internal Wave

Wave at the interface of water masses with different density. For example in the **Thermocline** where upper ocean warm water masses are separated from deeper colder waters.

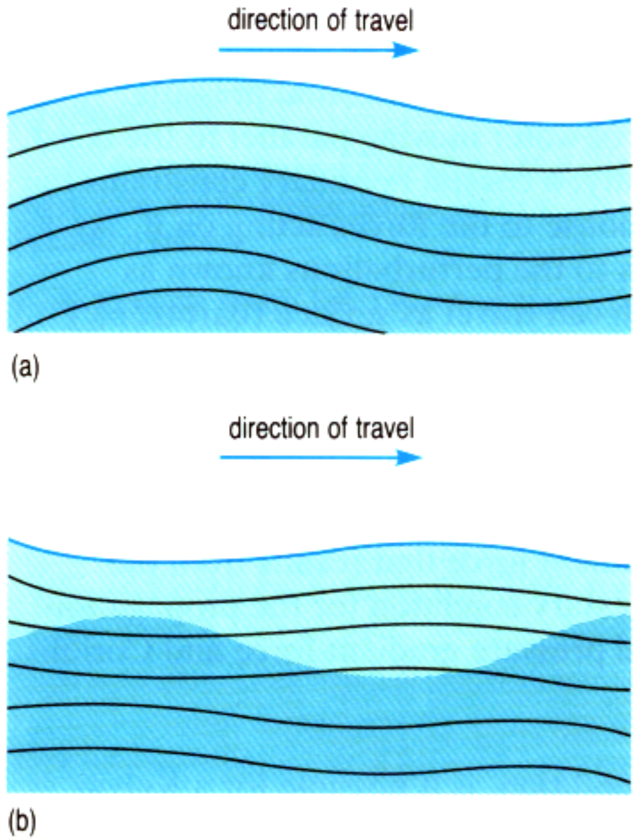


Figure 5.12 Examples of (a) a 'surface' long wave and (b) a long wave in the thermocline. In (a), the surface ocean as a whole moves up and down, and isobaric and isopycnic surfaces remain parallel. Such waves are therefore described as 'barotropic'. In (b), the passage of the wave changes the vertical density distribution, so that isopycnic surfaces are alternately compressed and separated. In addition, there are pressure variations over the surface of the density interface so that isobaric and isopycnic surfaces intersect; such waves are therefore described as 'baroclinic'.

Barotropic

Baroclinic

BAROTROPIC

Rossby radius of deformation external

$$\boxed{R = \frac{\sqrt{gH}}{f} = \frac{c}{f}}$$

Definition:

Distance that a particle or wave moving at a certain speed needs to cover in order to be affected by the rotation of the planet.

If *d* is the total depth of the water, we call this the BAROTROPIC DEFORMATION RADIUS or EXTERNAL RADIUS

BAROCLINIC

Rossby radius of deformation internal

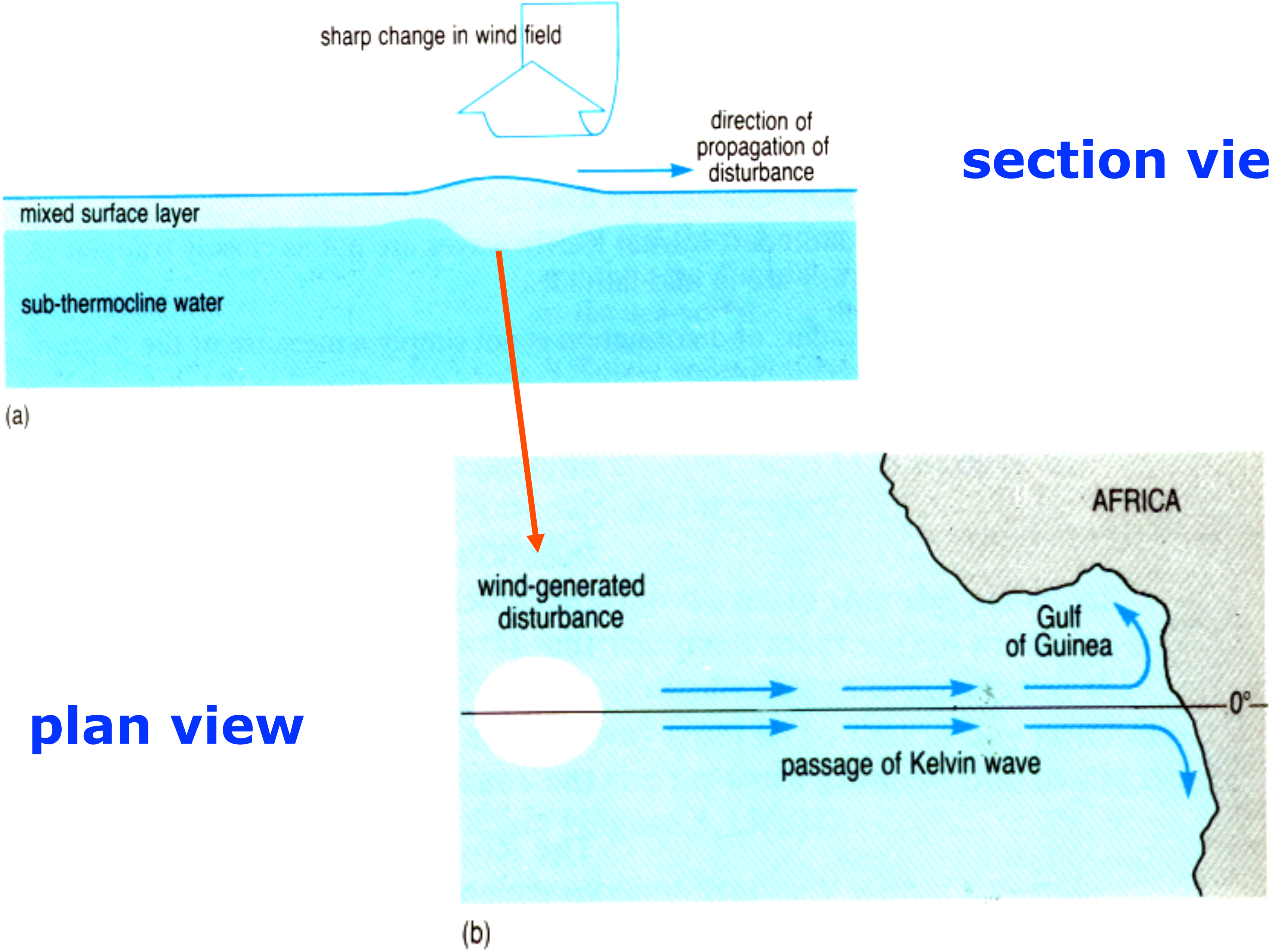
$$R = \frac{\sqrt{gH_{\text{upper layer}}}}{f} = \frac{c_{\text{upper layer}}}{f}$$

Definition:

Distance that a particle or wave moving at a certain speed needs to cover in order to be affected by the rotation of the planet.

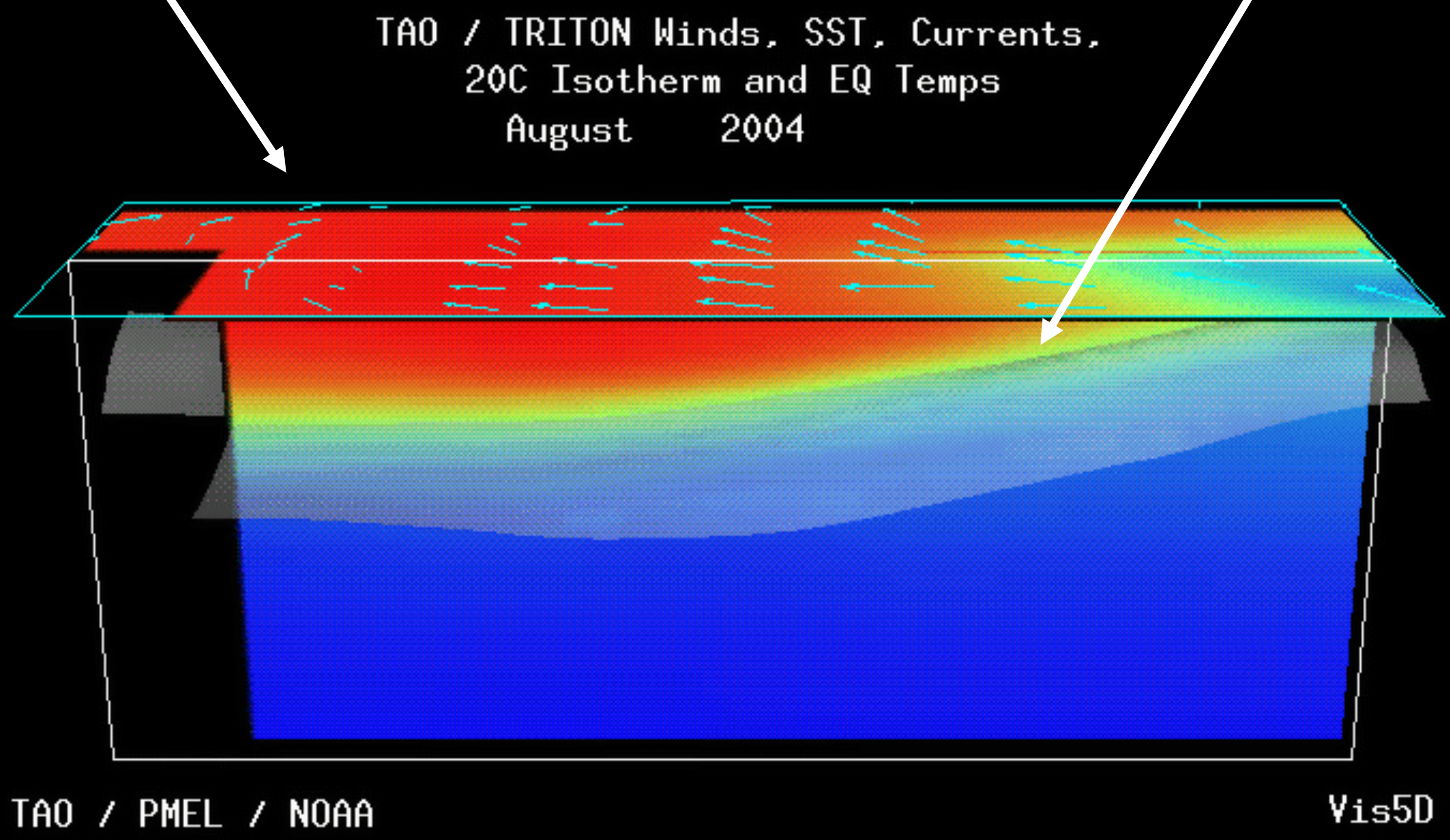
If *d* is the depth of the upper ocean layer, we call this the BAROCLINIC DEFORMATION RADIUS or INTERNAL RADIUS

An example of Equatorial Kelvin Wave



section view

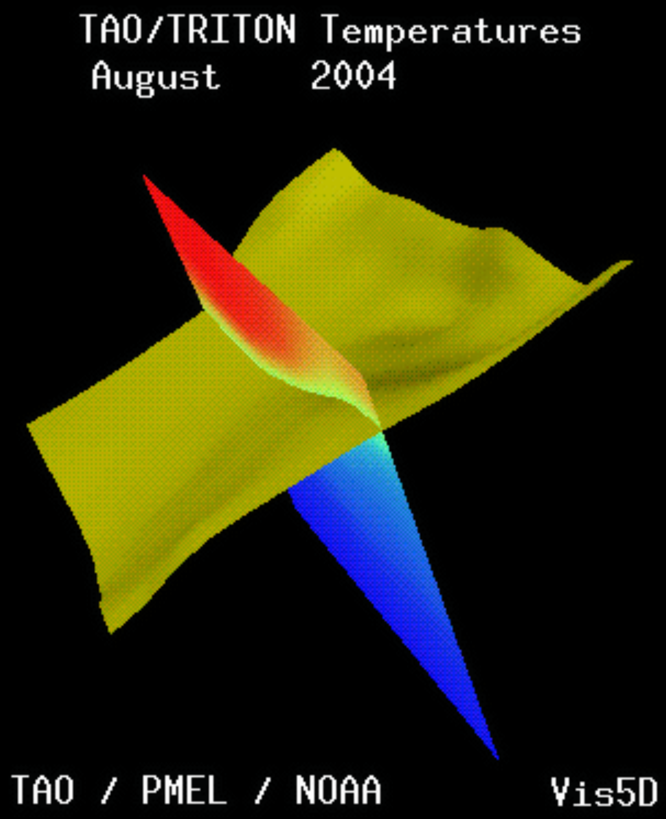
plan view

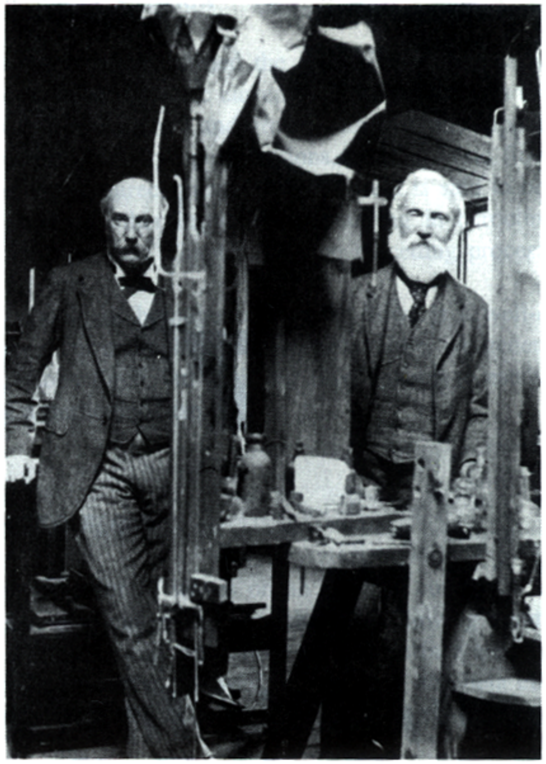


Western Pacific Warm pool

Equatorial Thermocline

Equatorial Thermocline





William Thomson, Lord Kelvin

1824 – 1907

(*Standing at right, in laboratory of Lord Rayleigh, left*)

Named professor of natural philosophy at the University of Glasgow, Scotland, at age 22, William Thomson became quickly regarded as the leading inventor and scientist of his time. In 1892, he was named Baron Kelvin of Largs for his technological and theoretical contributions leading to the successful laying of the transatlantic cable. A friend of Joule's, he helped establish a firm theory of thermodynamics and first defined the absolute scale of temperature. He also made major contributions to the study of heat engines. With Hermann von Helmholtz, he estimated the ages of the earth and sun and ventured in fluid mechanics (see Figures 11-2 and 11-3). His theory of the so-called Kelvin wave was published in 1879 (under the name William Thomson). His more than 300 original papers left hardly any aspect of science untouched. He is quoted as saying that he could understand nothing of which he could not make a model. (*Photo by A. G. Webster.*)

Inertia-gravity waves (Poincare' waves)

Governing Equations:

$$\begin{aligned}
\frac{\partial u}{\partial t} - f v &= - g\,\frac{\partial \eta}{\partial x} \\
\frac{\partial v}{\partial t} + f u &= - g\,\frac{\partial \eta}{\partial y} \\
\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0
\end{aligned}$$

Assume solutions in the form:

$$\begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = \Re \begin{pmatrix} A \\ U \\ V \end{pmatrix} e^{\mathrm{i}\,(k_x x + k_y y - \omega t)}$$

Wave numbers

Frequency

Substitution of solution into equations leads to:

$$\begin{aligned}
-\,\mathrm{i}\omega U - f V &= -\,\mathrm{i}\,g k_x A \\
-\,\mathrm{i}\omega V + f U &= -\,\mathrm{i}\,g k_y A \\
-\,\mathrm{i}\omega A + H\,(\mathrm{i}\,k_x U + \mathrm{i}\,k_y V) &= 0.
\end{aligned}$$

The determinant of this linear system vanishes if:

$$\boxed{\omega\,[\omega^2 - f^2 - gH\,(k_x^2 + k_y^2)] = 0}$$

dispersion relationship

Inertia-gravity waves (Poincare' waves)

Governing Equations:

$$
\begin{aligned}
\frac{\partial u}{\partial t} - f v &= -g \frac{\partial \eta}{\partial x} \\
\frac{\partial v}{\partial t} + f u &= -g \frac{\partial \eta}{\partial y} \\
\frac{\partial \eta}{\partial t} + H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0
\end{aligned}
$$

Assume solutions in the form:

$$
\begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = \Re \begin{pmatrix} A \\ U \\ V \end{pmatrix} e^{\mathrm{i}\,(k_x x + k_y y - \omega t)}
$$

Wave numbers

Frequency

Substitution of solution into equations leads to:

$$
\begin{aligned}
-\,\mathrm{i}\omega U - f V &= -\,\mathrm{i}\, g k_x A \\
-\,\mathrm{i}\omega V + f U &= -\,\mathrm{i}\, g k_y A \\
-\,\mathrm{i}\omega A + H\,(\mathrm{i}\, k_x U + \mathrm{i}\, k_y V) &= 0.
\end{aligned}
$$

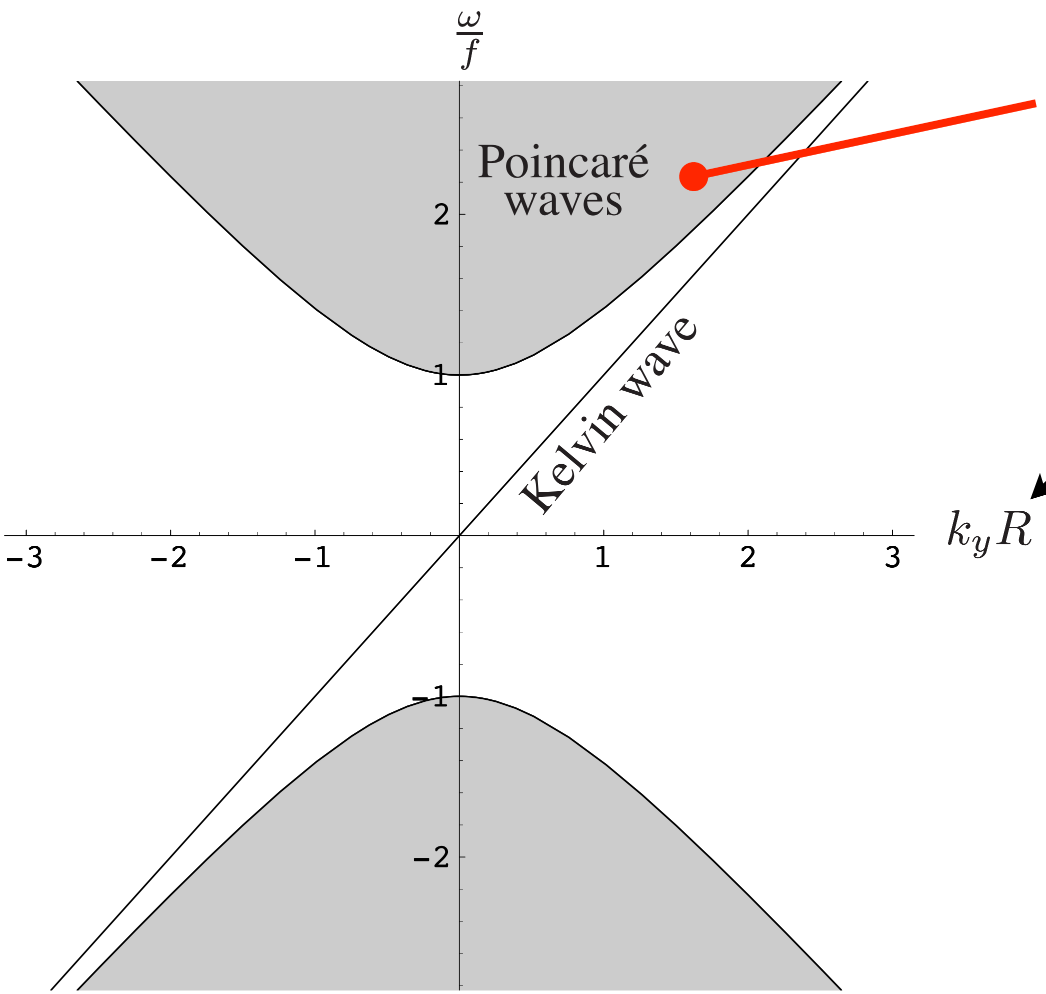
dispersion relationship

$$
\boxed{
\begin{aligned}
\omega &= 0 \\
\omega &= \sqrt{f^2 + gH\,k^2}
\end{aligned}
}
$$

with:

$$k = (k_x^2 + k_y^2)^{1/2}$$

Inertia-gravity waves (Poincare' waves)



superinertial frequency

$$R = \frac{\sqrt{gH}}{f} = \frac{c}{f}$$

Figure 9-3 Recapitulation of the dispersion relation of Kelvin and Poincaré waves on the f-plane and on a flat bottom. While Poincaré waves (gray shades) can travel in all directions and occupy therefore a continuous spectrum in terms of k_y, the Kelvin wave (diagonal line) propagates only along a boundary.

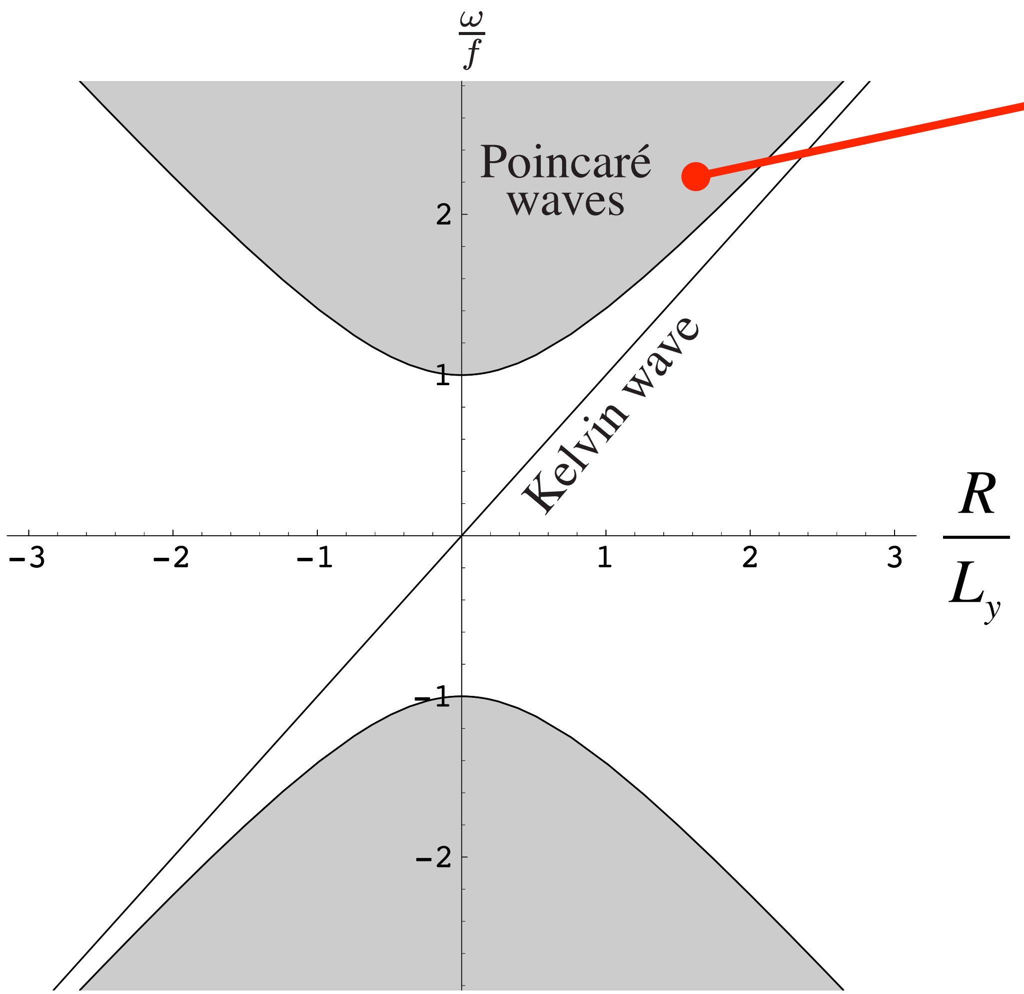
dispersion relationship

$$\boxed{\begin{aligned} \omega &= 0 \\ \omega &= \sqrt{f^2 + gH\,k^2} \end{aligned}}$$

with:

$$k = (k_x^2 + k_y^2)^{1/2}$$

Inertia-gravity waves (Poincare' waves)



superinertial frequency

$$R = \frac{\sqrt{gH}}{f} = \frac{c}{f}$$

1) inertial waves

$$L_y >> R \quad \rightarrow \quad \omega \simeq f$$

2) gravity waves

$$L_y << R \quad \rightarrow \quad \omega \simeq \sqrt{gH}\,k$$

dispersion relationship

$$\boxed{\begin{aligned} \omega &= 0 \\ \omega &= \sqrt{f^2 + gH\,k^2} \end{aligned}}$$

with:

$$k = (k_x^2 + k_y^2)^{1/2}$$

Planetary waves (Rossby waves)

Governing Equations:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Assume that geostrophic steady state undergoes slow evolution in time

This happens if we consider planetary effects

$$f = f_0 + \beta_0 y$$

$$f = 2\Omega \sin\varphi_0 + 2\Omega\frac{y}{a}\cos\varphi_0$$

beta parameter β_0

Planetary waves (Rossby waves)

Governing Equations for planetary waves:

$$
\begin{aligned}
\frac{\partial u}{\partial t} - (f_0 + \beta_0 y)v &= -g\frac{\partial \eta}{\partial x}\\
\frac{\partial v}{\partial t} + (f_0 + \beta_0 y)u &= -g\frac{\partial \eta}{\partial y}\\
\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0,
\end{aligned}
$$

Geostrophic balance dominates

$$
\begin{aligned}
v &\simeq +(g/f_0)\partial\eta/\partial x\\
u &\simeq -(g/f_0)\partial\eta/\partial y
\end{aligned}
$$

$$
\begin{aligned}
-\frac{g}{f_0}\frac{\partial^2\eta}{\partial y\partial t} - f_0 v - \frac{\beta_0 g}{f_0}y\frac{\partial\eta}{\partial x} &= -g\frac{\partial\eta}{\partial x}\\
+\frac{g}{f_0}\frac{\partial^2\eta}{\partial x\partial t} + f_0 u - \frac{\beta_0 g}{f_0}y\frac{\partial\eta}{\partial y} &= -g\frac{\partial\eta}{\partial y}
\end{aligned}
$$

Planetary waves (Rossby waves)

Governing Equations for planetary waves:

$$\begin{aligned}
\frac{\partial u}{\partial t} - (f_0 + \beta_0 y)v &= -g\frac{\partial \eta}{\partial x} \\
\frac{\partial v}{\partial t} + (f_0 + \beta_0 y)u &= -g\frac{\partial \eta}{\partial y} \\
\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0,
\end{aligned}$$

Geostrophic balance dominates

$$\begin{aligned}
v &\simeq +(g/f_0)\partial\eta/\partial x \\
u &\simeq -(g/f_0)\partial\eta/\partial y
\end{aligned}$$

substitute in continuity equation

solve for u and v

$$\begin{aligned}
u &= -\frac{g}{f_0}\frac{\partial \eta}{\partial y} - \frac{g}{f_0^2}\frac{\partial^2 \eta}{\partial x \partial t} + \frac{\beta_0 g}{f_0^2} y \frac{\partial \eta}{\partial y} \\
v &= \underbrace{+\frac{g}{f_0}\frac{\partial \eta}{\partial x}}_{\text{Geostrophic}} \underbrace{- \frac{g}{f_0^2}\frac{\partial^2 \eta}{\partial y \partial t} - \frac{\beta_0 g}{f_0^2} y \frac{\partial \eta}{\partial x}}_{\text{Ageostrophic}} .
\end{aligned}$$

Planetary waves (Rossby waves)

Governing Equations for planetary waves:

$$\begin{aligned}
\frac{\partial u}{\partial t} - (f_0 + \beta_0 y)v &= -g\frac{\partial \eta}{\partial x} \\
\frac{\partial v}{\partial t} + (f_0 + \beta_0 y)u &= -g\frac{\partial \eta}{\partial y} \\
\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0,
\end{aligned}$$

Geostrophic balance dominates

$$\begin{aligned}
v &\simeq +(g/f_0)\partial\eta/\partial x \\
u &\simeq -(g/f_0)\partial\eta/\partial y
\end{aligned}$$

one equation for free surface

$$\frac{\partial \eta}{\partial t} - R^2\frac{\partial}{\partial t}\nabla^2\eta - \beta_0 R^2\frac{\partial \eta}{\partial x} = 0$$

$$R = \sqrt{gH}/f_0$$

Planetary waves (Rossby waves)

assume solution in the form: $\cos(k_x x + k_y y - \omega t)$

one equation for free surface

$$\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t} \nabla^2 \eta - \beta_0 R^2 \frac{\partial \eta}{\partial x} = 0$$

$$R = \sqrt{gH} / f_0$$

Planetary waves (Rossby waves)

$$\omega = -\beta_0 R^2 \frac{k_x}{1 + R^2 (k_x^2 + k_y^2)}$$

dispersion relationship

assume solution in the form: $\cos(k_x x + k_y y - \omega t)$

one equation for free surface

$$\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t} \nabla^2 \eta - \beta_0 R^2 \frac{\partial \eta}{\partial x} = 0$$

$$R = \sqrt{gH} / f_0$$

Planetary waves (Rossby waves)

$$\boxed{\omega \;=\; -\;\beta_0 R^2\,\frac{k_x}{1\;+\;R^2\,(k_x^2\;+\;k_y^2)}}$$

$$f \;=\; f_0 \;+\; \beta_0 y$$

dispersion relationship

$$c_x \;=\; \frac{\omega}{k_x} \;=\; \frac{-\beta_0 R^2}{1\;+\;R^2\,(k_x^2\;+\;k_y^2)}$$

Planetary waves (Rossby waves) and topographic waves

$$\boxed{\omega = -\beta_0 R^2 \frac{k_x}{1 + R^2 (k_x^2 + k_y^2)}}$$

$$f = f_0 + \beta_0 y$$

dispersion relationship

$$c_x = \frac{\omega}{k_x} = \frac{-\beta_0 R^2}{1 + R^2 (k_x^2 + k_y^2)}$$

$$\boxed{\omega = \frac{\alpha_0 g}{f} \frac{k_x}{1 + R^2 (k_x^2 + k_y^2)}}$$

$$H = H_0 + \alpha_0 y$$

dispersion relationship

$$c_x = \frac{\omega}{k_x} = \frac{\alpha_0 g}{f} \frac{1}{1 + R^2 (k_x^2 + k_y^2)}$$

Planetary waves (Rossby waves)

$$\boxed{\omega \;=\; -\;\beta_0 R^2\,\frac{k_x}{1\;+\;R^2\,(k_x^2\;+\;k_y^2)}}$$

dispersion relationship

$$c_x \;=\; \frac{\omega}{k_x} \;=\; \frac{-\beta_0 R^2}{1\;+\;R^2\,(k_x^2\;+\;k_y^2)}$$

Shorter waves : $L \;\lesssim\; R, \quad \omega \;\sim\; \beta_0 L$

Longer waves : $L \;\gtrsim\; R, \quad \omega \sim \dfrac{\beta_0 R^2}{L} \;\lesssim\; \beta_0 L.$

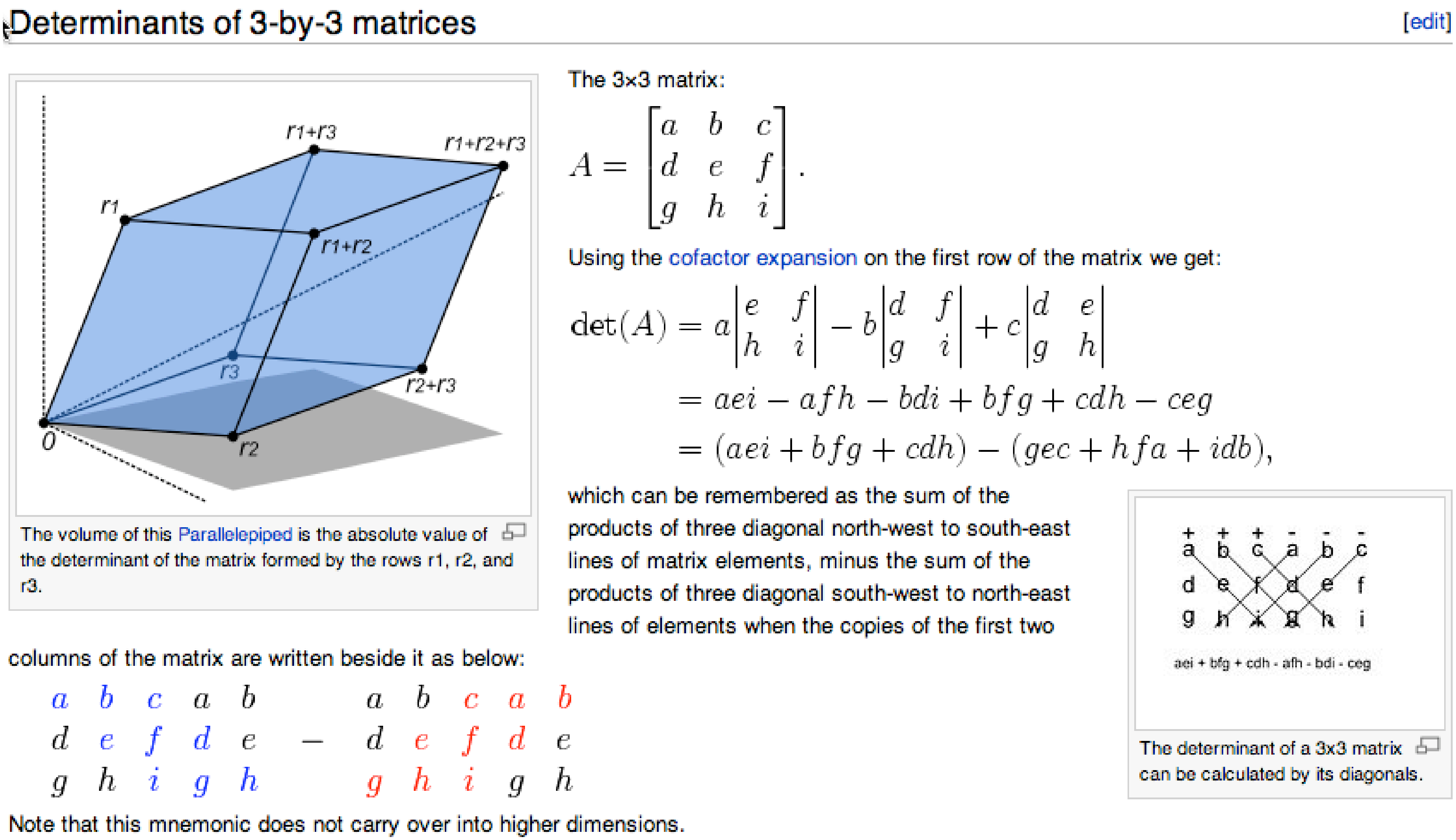
Planetary waves (Rossby waves)

dispersion relationship

$$\omega = -\beta_0 R^2 \frac{k_x}{1 + R^2 (k_x^2 + k_y^2)}$$

$$R = \sqrt{gH} / f_0$$

How to compute the determinant of a matrix



Determinants of 3-by-3 matrices

[edit]

The volume of this Parallelepiped is the absolute value of the determinant of the matrix formed by the rows r1, r2, and r3.

The 3×3 matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Using the cofactor expansion on the first row of the matrix we get:

$$\begin{aligned} \det(A) &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= aei - afh - bdi + bfg + cdh - ceg \\ &= (aei + bfg + cdh) - (gec + hfa + idb), \end{aligned}$$

which can be remembered as the sum of the products of three diagonal north-west to south-east lines of matrix elements, minus the sum of the products of three diagonal south-west to north-east lines of elements when the copies of the first two columns of the matrix are written beside it as below:

$$\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix} \quad - \quad \begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$$

Note that this mnemonic does not carry over into higher dimensions.

The determinant of a 3x3 matrix can be calculated by its diagonals.

Carl-Gustaf Arvid Rossby

1898 – 1957

A Swedish meteorologist, Carl-Gustaf Rossby is credited with most of the fundamental principles on which geophysical fluid dynamics rests. Among other contributions, he left us the concepts of planetary waves (Chapter 6), radius of deformation (Chapter 6), and geostrophic adjustment (Chapter 12). However, the dimensionless number that now bears his name was first introduced by the Soviet scientist I. A. Kibel' in 1940. Inspiring to young scientists, whose company he constantly sought, Rossby viewed scientific research as an adventure and a challenge. His accomplishments are marked by a broad scope and what he liked to call the *heuristic approach*, that is, the search for a useful answer without unnecessary complications. During a number of years spent in the United States, he established the meteorology departments at MIT and the University of Chicago. He later returned to his native Sweden to become the director of the Institute of Meteorology in Stockholm. (*Photo courtesy of Harriet Woodcock.*)