Predictor-corrector methods

when considering nonlinear source terms

$$\frac{du}{dt} = Q(t, u).$$

For simplicity, we consider here a scalar variable u, but extension to a $\mathbf{x} = (u, v)$, is straightforward.

The previous methods can be recapitulated as follows:

- The explicit Euler method (*forward scheme*):

$$\tilde{u}^{n+1} = \tilde{u}^n + \Delta t\, Q^n$$

- The implicit Euler method (*backward scheme*):

$$\tilde{u}^{n+1} = \tilde{u}^n + \Delta t\, Q^{n+1}$$

- The semi-implicit Euler scheme (*trapezoidal scheme*):

$$\tilde{u}^{n+1} = \tilde{u}^n + \frac{\Delta t}{2}\left(Q^n + Q^{n+1}\right)$$

- A general two-points scheme (with $0 \leq \alpha \leq 1$):

$$\tilde{u}^{n+1} = \tilde{u}^n + \Delta t\left[(1-\alpha)Q^n + \alpha Q^{n+1}\right]$$

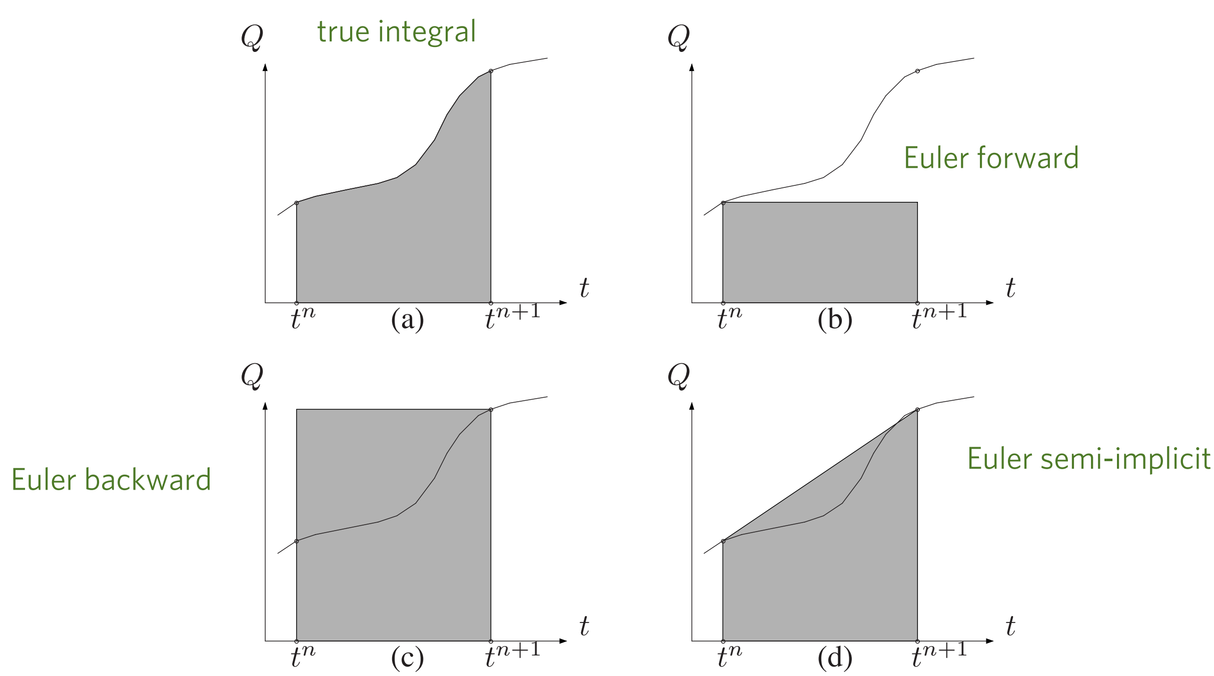


Figure 2-12 Time integration of the source term Q between t^n and t^{n+1}: (a) exact integration, (b) explicit scheme, (c) implicit scheme, and (d) semi-implicit, trapezoidal scheme.

scheme can be viewed as approximations of this integral:

$$u(t^{n+1}) = u(t^n) + \int_{t^n}^{t^{n+1}} Q\, dt,$$

all 2-point methods are all first order, except trapezoidal

Higher order methods require higher density in sampling Q

$$\frac{du}{dt} = Q(t, u).$$

however, Q depends on u

so how to know Q(n+1) without u(n+1)?

Such an approximation may proceed by using a first guess $\tilde{u}^\star$ in the Q term:

$$Q^{n+1} \simeq Q(t^{n+1}, \tilde{u}^\star), \tag{2.49}$$

as long as $\tilde{u}^\star$ is a sufficiently good estimate of $\tilde{u}^{n+1}$. The closer $\tilde{u}^\star$ is to $\tilde{u}^{n+1}$, the more faithful is the scheme to the ideal implicit value. If this estimate $\tilde{u}^\star$ is provided by a preliminary explicit (forward) step, according to:

2-step method

$$\begin{aligned} \tilde{u}^\star &= \tilde{u}^n + \Delta t\, Q(t^n, \tilde{u}^n) \\ \tilde{u}^{n+1} &= \tilde{u}^n + \frac{\Delta t}{2}\left(Q(t^n, \tilde{u}^n) + Q(t^{n+1}, \tilde{u}^\star)\right) \end{aligned}$$

forward step to guess u(n+1) --> u*

semi-implicit step

we obtain a two-step algorithm, called the *Heun method*. It can be shown to be second-order accurate.

This second-order method is actually a particular member of a family of so-called *predictor-corrector methods*, in which a first guess $\tilde{u}^\star$ is used as a proxy of $\tilde{u}^{n+1}$ in the computation of complicated terms.

Family of predictor-corrector methods --> 2nd order

Higher-order schemes

Need more values of Q

Option A: include intermediate points between tn tn+1 — Range-Kutta methods

Option B: or use values of Q(n-1) Q(n-2) etc.. — multi-steps methods

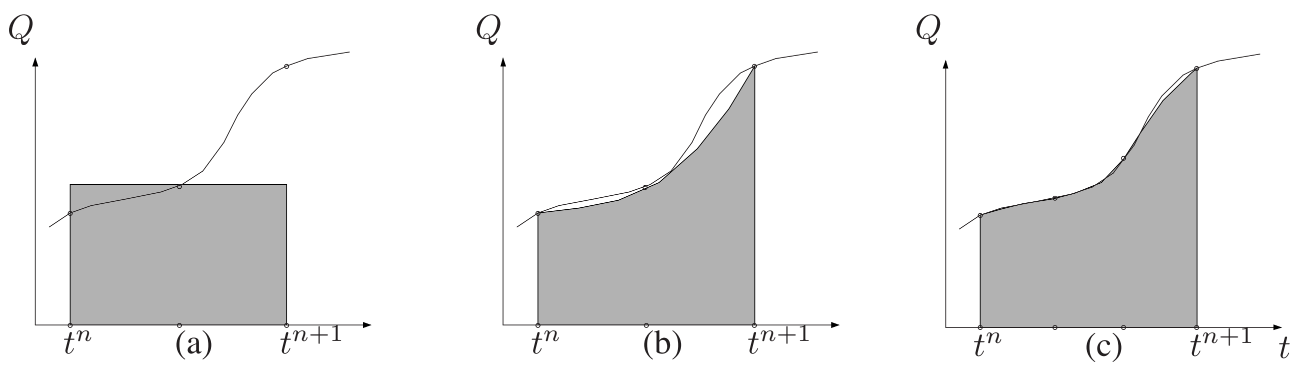


Figure 2-13 Runge-Kutta schemes of increasing complexity: (a) mid-point integration, (b) integration with parabolic interpolation, (c) with cubic interpolation.

Option A

Example (a) of mid-point rule

2nd order, no advantage over the Heun Scheme

$$\tilde{u}^{n+1/2} = \tilde{u}^n + \frac{\Delta t}{2} Q(t^n, \tilde{u}^n)$$

$$\boxed{\tilde{u}^{n+1} = \tilde{u}^n + \Delta t\, Q(t^{n+1/2}, \tilde{u}^{n+1/2}).}$$

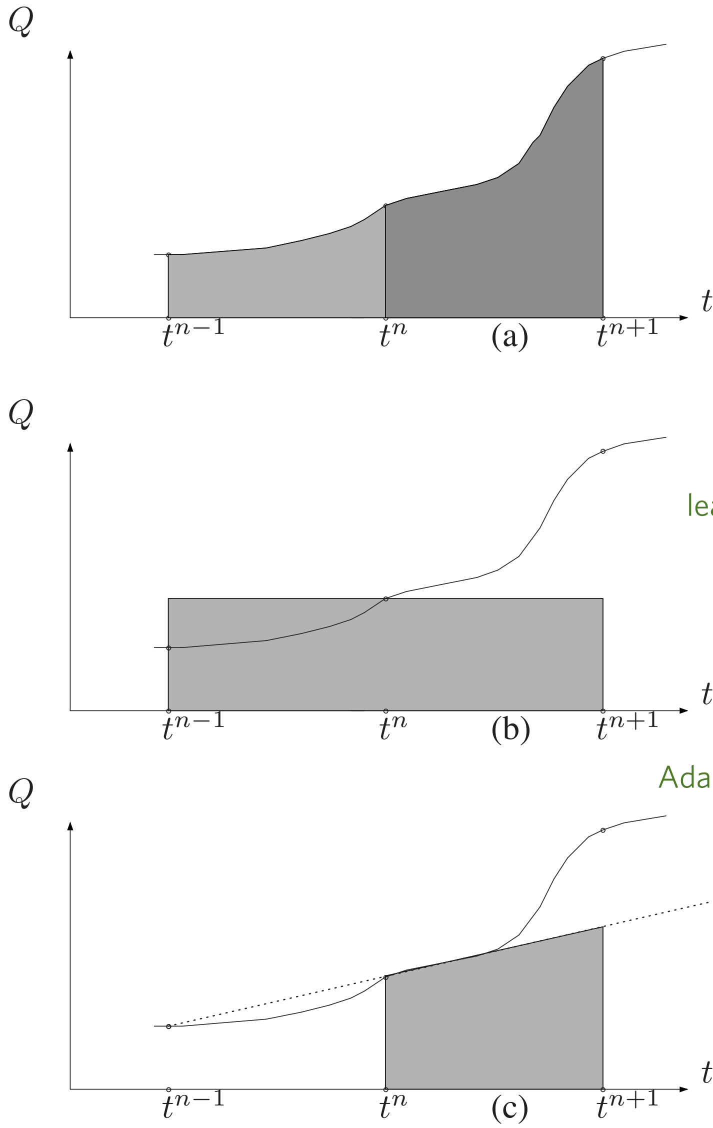
Example (b) parabolic integration

4th order, higher order achieved with fitting higher order polynbomials

$$\begin{aligned}
\tilde{u}_a^{n+1/2} &= \tilde{u}^n + \frac{\Delta t}{2} Q(t^n, \tilde{u}^n) \\
\tilde{u}_b^{n+1/2} &= \tilde{u}^n + \frac{\Delta t}{2} Q(t^{n+1/2}, \tilde{u}_a^{n+1/2}) \\
\tilde{u}^{\star} &= \tilde{u}^n + \Delta t\, Q(t^{n+1/2}, \tilde{u}_b^{n+1/2})
\end{aligned}$$

$$\boxed{\begin{aligned}
\tilde{u}^{n+1} &= \tilde{u}^n + \Delta t \left(\frac{1}{6} Q(t^n, \tilde{u}^n) + \frac{2}{6} Q(t^{n+1/2}, \tilde{u}_a^{n+1/2}) \right. \\
&\quad \left. + \frac{2}{6} Q(t^{n+1/2}, \tilde{u}_b^{n+1/2}) + \frac{1}{6} Q(t^{n+1}, \tilde{u}^{\star}) \right).
\end{aligned}}$$

Option B: using previous values of Q



leapfrog

Adam Bashforth

Figure 2-14 (a) Exact integration from t^{n-1} or t^n towards t^{n+1}, (b) leapfrog integration starts from t^{n-1} to reach t^{n+1}, whereas (c) Adams-Bashforth integration starts from t^n to reach t^{n+1}, using previous values to extrapolate Q over the integration interval t^n, t^{n+1}.

leapfrog 2nd order

The most popular method in GFD models is the *leapfrog method*, which simply reuses the value at time step $n-1$ to "jump over" the Q term at t^n in a $2\Delta t$ step:

$$\tilde{u}^{n+1} = \tilde{u}^{n-1} + 2\Delta t\, Q^n. \tag{2.53}$$

This algorithm offers second-order accuracy while being fully explicit.

Adam Bashforth 2nd order

$$\tilde{u}^{n+1} = \tilde{u}^{n} + \Delta t \frac{\left(3Q^{n} - Q^{n-1}\right)}{2},$$

these methods are typically harder to start because you are missing previous values of the function

Some considerations (read book)

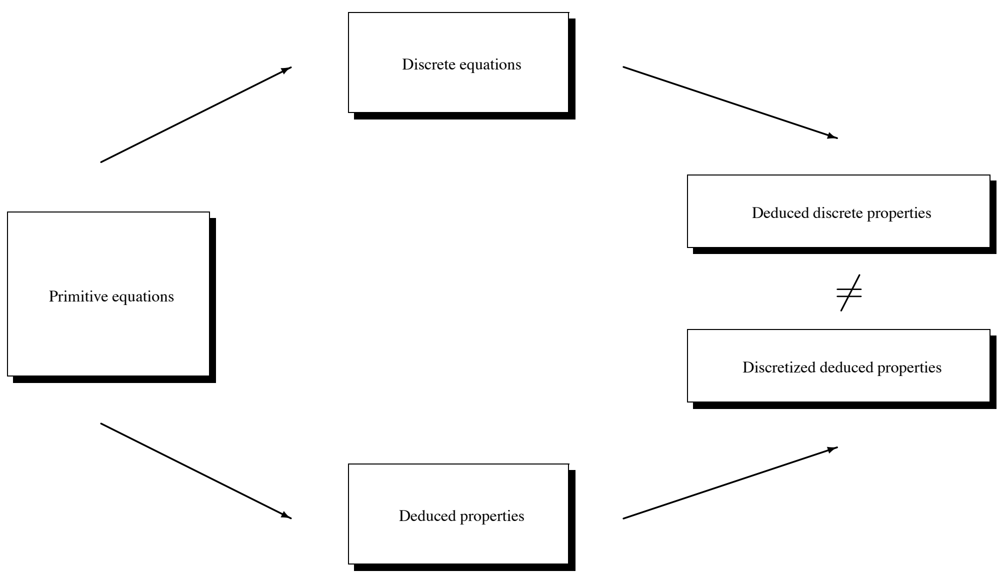


Figure 2-15 Schematic representation of discretization properties and mathematical properties interplay.

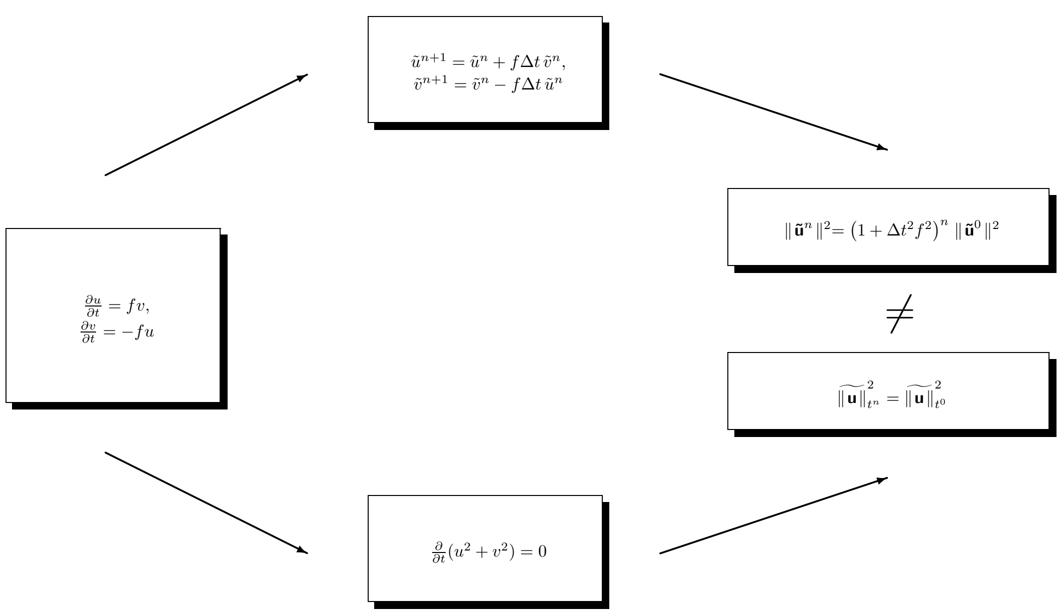


Figure 2-16 Schematic representation of discretization properties and mathematical properties interplay exemplified in the case of inertial oscillation.