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DYNAMICS OF STEADY OCEAN CURRENTS IN THE LIGHT OF EXPERIMENTAL FLUID MECHANICS

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INTRODUCTION

The present investigation may be regarded as a part of a systematic effort to introduce into meteorology and physical oceanography methods and results which for a number of years have contributed to the rapid growth and increasing practical significance of experimental fluid mechanics. This science has recognized that the exact character of the forces controlling the motion of a turbulent fluid is not known and that consequently there is very little justification for a purely theoretical attack on problems of a practical character. For this reason fluid mechanics has been forced to develop a research technique all of its own, in which the theory is developed on the basis of experiments and then used to predict the behavior of fluids in cases which are not accessible to experimentation.

In oceanography it has long been regarded as an axiom that the movements of the water are controlled by three forces, the horizontal pressure gradient, the deflecting force, and the frictional force resulting from the relative motion of superimposed strata. It is significant that thirty-five years of intensive theoretical work on this basis have failed to produce a theory capable of explaining the major features of the observed oceanic circulation below the pure drift current layer.

The present investigation considers a force which has been completely disregarded by theoretical investigators although its existence has been admitted implicitly by practically everyone who has approached physical oceanography from the descriptive side, namely the frictional force resulting from large-scale horizontal mixing. The introduction of this force permits us to see how motion generated in the surface layers may be diffused and finally dissipated without recourse to doubtful frictional forces at the bottom of the ocean.

A great number of practical hydrodynamic investigations of the observed oceanic current systems consist mainly in velocity calculations with the aid of the circulation theorem. Without denying the great practical value of the circulation theorem, the present investigation endeavors to emphasize a fact which by this time should have been generally accepted but which it not always kept in mind, namely the impossibility of drawing any conclusions regarding the cause of oceanic motions from the ordinary routine application of the circulation theorem.

In the first part of the paper the principal imperfections of the present theory for the oceanic circulation are set forth. Frictional forces due to horizontal mixing are then introduced and the effect of the earth's rotation on the horizontal eddy velocities analyzed. Tollmien's theory for the mixing along the edges of a steady stream moving through a resting fluid is then discussed and certain experimental verifications are described. With the aid of a principle first stated by G. I. Taylor, Tollmien's results are applied to current systems subject to a deflecting force. Finally certain important modifications resulting from the stratification in the ocean are treated.

In the second part of the paper an attempt is made to trace the mixing between the Gulf Stream and its surroundings with the aid of the observed distribution of temperature, salinity and oxygen. The results of this qualitative analysis seem to bear out the theoretical predictions.

The theory set forth is utterly incomplete, and serious objections may be raised against the looseness of the reasoning on which it is based. Nevertheless, the author believes that it may serve as a useful working hypothesis, since its predictions refer to an idealized stratified ocean and not to a non-existent homogeneous medium.

The present paper is to a large extent the result of fruitful cooperation between a number of persons. Dr. H. Peters of the Massachusetts Institute of Technology not only carried out certain experimental tests of Tollmien's theory but also, in a number of discussions, directed my attention to various investigations bearing on the relative merits of the momentum transfer and the vorticity transfer theories.

Mr. C. O'D. Iselin of the Woods Hole Oceanographic Institution has contributed his vast knowledge of the hydrographic conditions in the North Atlantic. Without his active cooperation it would have been impossible to carry through the investigation to a point where it could be tested against observations. Several of the conclusions here derived from purely theoretical considerations have already been reached by Mr. Iselin from a study of the hydrographic data collected by the Woods Hole Oceanographic Institution.

Mr. H. R. Seiwell's investigations of the oxygen distribution in the North Atlantic have been particularly helpful and are responsible for the choice of oxygen as an indicator of horizontal mixing.

The author is indebted to Dr. H. B. Bigelow for various helpful suggestions.

A brief account of the principal theoretical results presented below was given before the annual meeting of the Institute of the Aeronautical Sciences in New York, January, 1936.

At the date of writing this introduction, Dr. A. E. Parr of Yale University informs me that he has been led to conclusions of substantially the same nature as some of the ones here presented, through a study of recent hydrographic data from the Caribbean. Dr. Parr's results will be published in *Journal du Conseil.*

Woods Hole, July 15, 1936

I. THEORETICAL DISCUSSION

A. Formulation of Problem

Anybody who has attempted to construct, for his own satisfaction, theoretical working models of the permanent current system in the ocean or in the air or of some of the apparently steady phenomena of the secondary circulation, sooner or later runs up against the apparent impossibility of finding forces capable of producing, in the interior of the media under consideration, horizontal convergence or divergence on a scale comparable to that which actually must occur in nature. In cyclonic regions, surface friction produces an easily observed transport of air across the isobars towards lower pressure. Since the gradient wind supposed to prevail at higher levels is very nearly free from divergence there is apparently no way in which the accumulating surface air may be removed. One would therefore expect a rapid decay or filling up of these low pressure systems. Nevertheless, particularly the occluded cyclones of higher latitudes and the hurricanes of lower latitudes often seem to be characterized by a condition of approximate dynamic equilibrium.

A similar problem appears in the interpretation of the horizontal circulation of the ocean. The permanent anticyclonic wind system of the North Atlantic Ocean produces a steady accumulation of surface water in lower latitudes and a corresponding slope of the sea surface. The resulting gradient current system should be very nearly free from horizontal divergence and thus incapable of re-establishing equilibrium. To avoid this difficulty Ekman,[1] in his general theory of the circulation of a homogeneous ocean, assumes that the bottom friction is so strong that it produces a divergence sufficient to offset the wind-produced surface convergence. It is easily shown that the bottom friction required for this purpose must be of the same order of magnitude as the surface friction.

Ekman's solution implies that bottom water and surface water are equivalent. In each region of surface convergence and bottom divergence there must be a descending motion, in each region of surface divergence and bottom convergence there must be an ascending motion, so that bottom and surface water continually replace each other. While this may be acceptable in the ideal case of a homogeneous ocean, it is in sharp disagreement with observed conditions in the real, stratified ocean.

According to Ekman's theory the equatorial side of the subtropical Highs must be characterized by such accumulation of surface water. The observed steep thermocline in these regions shows that the accumulation and sinking of surface water must cease within a depth of a few hundred meters, in contrast with the theoretical prediction, although there are definite indications that strong horizontal convergence and sinking must occur in these upper layers.

Since the vertical circulation does not extend all the way down it may be argued that the water, because of its stratification, has a cellular structure, each cell being separated through approximately horizontal surfaces of discontinuity from the cells above and below. Each boundary surface would then act as a "false" bottom and each cell would have a practically independent circulation. In order to have steady conditions and zero horizontal divergence in each cell, it would be necessary for the shearing stresses at each boundary to be of the same order of magnitude as those at the surface. This stress distribution would produce a much stronger circulation in the bottom cell than indicated by available data. Furthermore, for each new cell introduced the surface current velocity is raised, so that the suggested scheme most likely would produce impossible surface velocities.

It is no doubt possible to overcome some of these difficulties locally by considering the deviations from gradient flow associated with inertia forces. This is the line of attack followed by Ekman in his latest investigations. It is as yet impossible to estimate completely the extent to which this much needed extension of the theory will eliminate the difficulties listed above. However, in this connection the following comment is pertinent:

The horizontal circulation of the southern half of the North Atlantic may be represented as a gigantic *stationary* anticyclonic eddy maintained by the permanent anticyclonic wind system over the same area. Since the mean motion is steady, the mean total torque round a vertical axis must vanish. In Ekman's theory this is accomplished through the introduction of frictional forces at the bottom, the torque of which balances the wind torque. The consideration of inertia forces in no way removes the need for this balancing frictional force at the bottom. Actually observations indicate that the motion near the bottom is vanishingly small and thus incapable of producing frictional forces of any significance.

An inspection of a current chart for the North Atlantic indicates that strong eddying motion occurs at many places along the borders of the basin. *Thus it seems possible that the required balance may be established through frictional forces originating on the continental slopes and transmitted through the water as shearing stresses acting on vertical surfaces parallel to the horizontal current components.* For the sake of brevity shearing stresses of this type will here be referred to as lateral stresses, while the designation normal stresses is reserved for stresses acting on horizontal surfaces and produced by the vertical variation in horizontal velocity. It is evident, from a study of the relative horizontal and vertical dimensions of atmospheric and oceanic systems, that the lateral stresses must be many times larger than the normal ones if they are to be of any dynamic significance.

The idea that momentum may be transferred horizontally through turbulence is not new. In a much-discussed paper published in 1921 Defant[2] assumed that the travelling cyclones and anticyclones may be regarded as turbulent elements superimposed on the mean circulation of the atmosphere in middle latitudes. Defant used this conception of the general circulation to compute the advective transfer of heat from the equator to the poles. However, in Defant's case the eddying components are quite large compared to the mean motion, so large, in fact, that the mean motion of the air is often completely obscured by the presence of the eddying motion. It is doubtful that these large eddies derive their energy from the mean motion, and perhaps more likely that the reverse is true. Thus it appears desirable to select for study a steady fluid system characterized by a well-established primary mean motion and to determine the rôle played by lateral shearing stresses in the dynamics of this system.

Richardson and Proctor[3] have investigated horizontal diffusion in atmospheric currents by means of the scattering of volcanic ash and the scattering of small toy balloons. For distances ranging between 3 km. and 86 km. these authors obtained values of the horizontal diffusivity varying between $2 \cdot 10^6$ and $1.3 \cdot 10^9$ cm.^{2}/sec. It is reasonable to assume that the turbulent mechanism responsible for this scattering must produce an equally intensive lateral diffusion of horizontal momentum. The lateral stresses introduced above are simply a measure of this lateral eddy transport of momentum.

If Richardson's and Proctor's coefficients are expressed as eddy-viscosities, they range from $2.5 \cdot 10^3$ to $1.6 \cdot 10^6$ grams/cm.sec. Thus they are intermediate in magnitude between the values obtained from the study of vertical wind gradients, 10^2 grams/cm.sec., and the values obtained by Defant from an analysis of the general circulation as a turbulent process, 10^8 grams/cm.sec. In Defant's analysis of the general circulation the individual turbulent elements are supposed to consist of travelling cyclones and anticyclones or, more properly, of large bodies of air from different source regions. The diffusion process measured by Richardson and Proctor, and studied from another point of view in the present paper, deals with phenomena within a single air or ocean current and along its boundaries. It presupposes the existence of eddies whose dimensions must be measured in fractions of a kilometer up to, perhaps, twenty or thirty kilometers. The remarkable uniformity in air mass characteristics so often observed in our aerological data suggests that horizontal diffusion on such a large scale must occur with great regularity in the atmosphere. It is rather surprising then to find that the dynamic consequences of this horizontal diffusion mechanism never have been investigated.

Before proceeding, it may be worth while to point out how lateral shearing stresses affect the horizontal divergence. On the northern hemisphere, steady, non-accelerated motion in the atmosphere or in the ocean is characterized by the fact that to a given horizontal force P there corresponds a horizontal momentum M directed 90° to the right from P and having the value

$$M = \frac{P}{2\omega \sin L}, \tag{1}$$

where L is the latitude and ω is the angular velocity of the earth. As an illustration, consider a vertical air column in a field of straight, parallel isobars. This column is acted upon by the horizontal pressure gradient and by the frictional force between the air and the ground. It is evident that the component of its momentum across the isobars must correspond to the component of ground friction parallel to the isobars. If the same column of air is subject not only to normal stresses but also to suitable lateral shearing stresses, the resultant force along the isobar direction and thus also the total flow across the isobars may be made to vanish.

Because of the earth's rotation, the effect of the normal shearing stresses originating at a horizontal boundary vanish within a relatively short vertical distance. Outside these shallow boundary layers the velocities vary only slowly along the vertical, at least when there is steady motion and when the medium considered is in barotropic equilibrium; thus we are permitted to assume that the lateral shearing stresses are reasonably independent of the vertical coordinate through fairly deep strata. This effect of the earth's rotation simplifies a separation of the effects of lateral and normal stresses; such a separation, on the other hand, is not readily possible in the case of small-scale hydraulic experiments.

The balance of forces in a horizontal direction normal to the mean motion, which consists in an equilibrium between deflecting force and horizontal pressure gradient, is not materially affected by the presence of lateral stresses. *This balance, which for the atmosphere takes the form of the ordinary gradient wind relationship and which is also utilized for so called "dynamic velocity calculations" of ocean currents, does not prescribe a definite velocity profile across the current.* More specifically, if we consider the mass distribution in a certain vertical plane, it is always possible to find a distribution of velocities normal to this plane such that the resulting deflecting force everywhere balances the horizontal pressure gradient resulting from the mass distribution (distribution of solenoids). Conversely, it is always possible to find a mass distribution in a vertical plane such that the resulting horizontal pressure gradient balances the deflecting force associated with an arbitrary distribution of velocities normal to the plane.

On the other hand, the effect of lateral stresses acting in the direction of the motion must be to produce certain characteristic transversal velocity profiles. If, then, through an analysis of available observations, the existence of certain preferred atmospheric or oceanic current profiles is established, which profiles from a comparison with completely controlled laboratory experiments appear to be the result of frictional forces (lateral stresses), we are reasonably justified in assuming that the associated mass (solenoid) distribution in a transversal plane must be regarded as a result rather than as a cause of the motion. This point is stressed here since there seems to be a tendency on the part of many oceanographers to regard the mass distribution, which serves as a starting point in all dynamic calculations of so-called "convection" currents, as their cause. As a matter of fact, it is easy enough to show how, on a rotating globe, solenoids may be generated through mechanical means.[4] It is possible to develop criteria for the separation of such secondary *dynamic* solenoids from the *thermal* solenoids, which are the ultimate cause of all motion in the atmosphere. Thus one should expect to find the vertical correlation curve between temperature and salinity to be independent of location in an ocean current section whose solenoids are dynamic in origin. Similarly, in a section across a steady air current in which the solenoids are of secondary character, the vertical correlation between specific humidity and potential temperature ought to be reasonably constant. Illustrative examples will be furnished in the second part of this investigation.

B. Effect of the Earth's Rotation on Lateral Stresses

The evaluation of lateral stresses in the air or in the ocean brings up another problem of general significance, namely, the effect of curvature and of the earth's rotation on the turbulent exchange of momentum between fluid strata moving side by side. It thus forces us to choose between the "vorticity-transport" theory developed by Taylor[5] and the "momentum-transport" theory developed by Prandtl.[6] Taylor has pointed out that the structure of straight fluid current systems may be interpreted equally well with the aid of the one as with the aid of the other of these two theories but that, in the case of curved flow or flow in rotating systems the two theories lead to mutually exclusive results. It seems appropriate to follow up this comment of Taylor's with an analysis of the predictions of the two theories in as far as atmospheric and oceanic motion is concerned.

As a starting point we choose a steady terrestrial fluid system rotating cyclonically relative to the surface of the earth around a certain vertical axis A. The rotation of the earth itself may be resolved into a rotation around A and a rotation around an axis normal thereto. The latter rotation is without significance in the present connection. The relative linear velocity at a distance r from the axis is given by v. It we designate by

$f = 2\omega \sin L$ the Coriolis parameter, it follows that the absolute linear velocity V around the axis has the value

(2) $$V = v + \tfrac{1}{2} f r.$$

The absolute angular momentum around A is given by

(3) $$\Omega = rV = rv + \tfrac{1}{2} f r^2.$$

According to the momentum transfer theory each element displaced along the radius tends to retain its original angular momentum. Thus an element displaced from r to $r+l$ will produce, in its new position, a deviation of the observed angular momentum from the mean, given by

(4) $$\Omega' = \Omega_r - \Omega_{r+l} = -l \frac{\partial \Omega}{\partial r},$$

and consequently a deviation of the tangential velocity from the mean, given by

(5) $$v' = -\frac{l}{r} \frac{\partial \Omega}{\partial r}.$$

Assuming equipartition of eddy energy it follows that the shearing stress is given by

(6) $$\tau = -\rho \, \overline{u'v'} = \rho \, \overline{u'l} \frac{1}{r} \frac{\partial \Omega}{\partial r} = \rho \frac{l^2}{r^2} \left(\frac{\partial \Omega}{\partial r} \right)^2.$$

In this expression u' represents the radial (eddying) velocity. Thus the momentum transfer theory indicates that the shearing stress vanishes when the absolute angular momentum is independent of the distance from the axis. One may now introduce the relative motion in the above expression. The result is

(7) $$\tau = \rho l^2 \left(\frac{\partial V}{\partial r} + \frac{V}{r} \right)^2 = \rho l^2 \left(\frac{\partial v}{\partial r} + \frac{v}{r} + f \right)^2.$$

If the radius of curvature is sufficiently large, the above expression reduces to the form

(8) $$\tau = \rho l^2 \left(\frac{\partial v}{\partial x} + f \right)^2,$$

where x is a horizontal coordinate counted positive in a direction 90° to the right of the direction in which the current is flowing. Thus the momentum transfer theory indicates that in a straight air or ocean current the lateral shearing stresses vanish when the velocity decreases towards the right edge of the current at the rate

(9) $$\frac{\partial v}{\partial x} = -f.$$

This is a very steep rate, which in middle latitudes (43°) corresponds to a rate of shear of 1 cm.p.s. in 100 meters. Such horizontal rates of shear are hardly ever observed in the ocean and in the atmosphere they occur only along fronts. Thus, according to the momentum transfer theory, the right edge of a current always tends to accelerate the left edge, even though the velocity to the right may be considerably less than the velocity to the left. In particular, a broad, uniform current would be subject to shearing stresses tending to produce a velocity profile with a steep drop in velocity towards the right of the current.

This result of the momentum transfer theory may be obtained in a different way, which brings out another discrepancy between the two theories. Consider a straight current flowing in the direction of the y-axis. There is equilibrium between the horizontal pressure gradient and the deflecting force corresponding to the mean velocity $\bar{v}$. In the course of the turbulent motion, individual elements will move across the stream. If the horizontal velocity components of the moving elements are designated by u and v, their equations of motion will be

$$\frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} = f(v-\bar{v}) \tag{10}$$

$$\frac{dv}{dt} = -fu. \tag{11}$$

The second of these two equations may be integrated at once and gives

$$v - v_0 = -f(x-x_0) = -fl, \tag{12}$$

where l means the displacement of the element cross-stream and the subscript o refers to the initial state. We may assume that the element originally had the same velocity downstream as its surroundings so that

$$v_0 = \bar{v}_0. \tag{13}$$

Thus, as a result of the displacement l, the element will appear in its new position with a velocity in excess of that of the surroundings. This excess is given by

$$v' = v - \bar{v}_l = \bar{v}_0 - fl - \bar{v}_l = -l\left(f + \frac{\partial \bar{v}}{\partial x}\right). \tag{14}$$

The lateral shearing stress may be computed from

$$\tau = -\rho\overline{u'v'} = \rho\overline{lu'}\left(f + \frac{\partial \bar{v}}{\partial x}\right). \tag{15}$$

Again it appears that the shearing stress disappears, not when the current is uniform, but when

$$\frac{\partial \bar{v}}{\partial x} = -f. \tag{16}$$

Furthermore, the rotation of the earth would appear to produce strong stabilizing forces tending to suppress turbulence. If we insert the expression

$$v = \bar{v}_0 - fl \tag{17}$$

in the first equation of motion (10), we find

$$\frac{du}{dt} = f(\bar{v}_0 - fl - \bar{v}_l) = -fl\left(f + \frac{\partial \bar{v}}{\partial x}\right). \tag{18}$$

Since the last factor is positive for practically all atmospheric or oceanic systems it follows that the acceleration du/dt is negative. Thus elements moving cross-stream are subject to a strong restoring force. Per unit mass and displacement this force has the value

$$RF = f\left(f + \frac{\partial \bar{v}}{\partial x}\right). \tag{19}$$

Richardson[7] has introduced a non-dimensional parameter (P) which may serve as a measure for the effectiveness of stabilizing forces in suppressing turbulence. It is obtained by dividing the stablizing force through the square of the vorticity. The significance of this quantity is that it measures the ratio between the amount of eddy energy lost through work against stabilizing forces and the amount of eddy energy produced through the work of the eddy shearing stresses. On the basis of the momentum transfer theory this ratio has the value

$$P = \frac{f\left(f + \frac{\partial \bar{v}}{\partial x}\right)}{\left(f + \frac{\partial \bar{v}}{\partial x}\right)^2} = \frac{f}{f + \frac{\partial \bar{v}}{\partial x}} \approx 1, \tag{20}$$

which is sufficiently high to suppress lateral turbulence in a very efficient way.

The momentum transfer theory, as applied to a symmetric rotating system, assumes that individual elements retain their original absolute angular momentum during radial displacements. Taylor (l.c.) has pointed out that this assumption implies that local pressure gradients resulting from the displacements can be neglected. This may not always be true. On the other hand, we do know that the elements in the absence of viscosity retain their original vorticity regardless of displacements. The expression for the frictional force should be such as to take cognizance of this fact. The effect of the eddies is to produce a transport of vorticity from regions of high to regions of low vorticity. Since the gradient of vorticity is not changed by a rotation of the system as a whole around a fixed axis, this rotation simply having the effect of adding a constant amount of vorticity to each point in the system, it appears that the shearing stresses must have such a form that the addition or subtraction of a solid rotation does not change their value. In the case of a radially-symmetric, terrestrial fluid system, rotating around a vertical axis, this is the case if we assume

$$\tau = \rho l^2 \left(\frac{\partial V}{\partial r} - \frac{V}{r}\right)^2, \tag{21}$$

where V is the absolute velocity. With the aid of (2) we may introduce the velocity v relative to the surface of the earth and obtain

$$\tau = \rho l^2 \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2. \tag{22}$$

For straight currents this expression reduces to

$$\tau = \rho l^2 \left(\frac{\partial v}{\partial x}\right)^2, \tag{23}$$

where x again is a horizontal coordinate pointing cross-stream to the right of the direction of flow. From this expression the reference to the rotation of the earth has disappeared.

Without attempting to attack the general question of the effect of local pressure gradients, it may be stated that the present problem offers a particularly good opportunity for its study. Let us assume that the lateral stresses in the ocean are caused by vertical fluid columns moving cross-stream in an irregular fashion. To study the effect of local pressure gradients, assume furthermore that each column has a circular cross-section with a radius a and is so deep that the motion of the surrounding displaced water is reasonably free from horizontal convergence or divergence. Under these assumptions the motion of the fluid outside the cylinder will be of the type known from potential theory, but the pressure distribution will differ from the one given by the classical solution. If the fluid at some distance from the cylinder is at rest or in uniform horizontal motion (relative to the earth) it is found that the moving cylinder is acted upon by two forces. The first is the deflecting force, which, per unit length of the cylinder, has the absolute value

$$DF = \rho' \pi a^2 f u. \tag{24}$$

In this expression ρ' is the density of the fluid in the cylinder and u is its velocity. This deflecting force is horizontal and directed 90° to the right from the velocity u. The second force results from the pressure distribution in the surrounding fluid and has the value

$$PF = \rho \pi a^2 f u, \tag{25}$$

where ρ is the density of the displaced fluid. This second force acts in the opposite direction to the first. If the two densities are equal, the two forces balance each other.

The creation of horizontal pressure gradients around the moving cylinder requires changes in level at the free surface and thus also horizontal divergence, but the amount of this divergence can be made negligibly small by making the cylinders sufficiently deep. Thus the motion of the cylinder is not affected by the rotation of the earth, contrary to the assumption underlying the momentum transfer theory.

C. "Coriolian" Pressure Gradients

Returning to the main topic we may say that in the absence of horizontal convergence and divergence, a moving fluid portion will be subjected to horizontal pressure gradients which will completely offset the deflecting force. This result was first obtained by Taylor (l. c., p. 696) in 1932 and expressed by him in the following fashion: "If ψ is the stream function at any instant of any two-dimensional motion of a viscous incompressible fluid, then the whole system may be rotated with uniform angular velocity Ω about an axis perpendicular to the plane of motion, and a motion relative to the rotating axes identical in every respect with the original motion is possible. If p is the pressure corresponding with the original motion, the pressure when the whole system is rotated is $p+2\rho\,\Omega\psi+\frac{1}{2}\rho\,\Omega^2 r^2$, where r is the distance from the centre of rotation. The stresses due to viscosity are unaltered by the rotation as also are the stresses due to turbulence."

In our case the term $\frac{1}{2}\rho\,\Omega^2 r^2$, obtained from the centrifugal force associated with the rotation of the coordinate system, drops out since the centrifugal force is offset by a component of the true acceleration of gravity. The quantity $2\,\Omega$ occurring in the above quotation from Taylor is identical with the Coriolis parameter f in our notation. Thus sufficiently deep currents may be analyzed as if the earth were not rotating, provided we subtract from the acting forces the secondary "Coriolian" pressure gradients which represent the reaction of the fluid to the rotation of the earth. These gradients are given by

$$-\frac{\partial p_c}{\partial x} = -\rho f v \tag{26}$$

$$-\frac{\partial p_c}{\partial y} = +\rho f u. \tag{27}$$

It is apparent that a pressure field satisfying these equations always can be found provided

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{28}$$

that is, provided the motion is free from horizontal convergence and divergence. The Coriolian pressure gradients are normally many times larger than the dynamically more significant residuals. This fact enables us to compute currents with a reasonable degree of accuracy from the total pressure gradients.

The preceding discussion should serve to emphasize the complex, and to a very large extent secondary, character of the horizontal pressure distribution. With respect to the oceanic circulation it seems particularly appropriate to emphasize the following point: The swift currents in the ocean troposphere owe their existence, directly or indirectly, to wind friction. In the case of such motions, momentum may be transferred from layer to layer through shearing stresses, and the pressure distribution may then be entirely secondary in character. A simple illustration is furnished by a fluid contained between two concentric vertical cylinders and having one free surface. If the outer cylinder is set in motion it will gradually transmit its momentum to deeper and deeper fluid strata through shearing stresses. A radial pressure gradient (sloping free surface) gradually develops as a reaction to the centrifugal force but plays no rôle in the transfer of momentum.

Thus, in the case of terrestrial systems which are free from convergence or divergence, it is possible to eliminate the influence of the rotation of the earth through the balance between deflecting force and the Coriolian pressure field. The remaining terms, which have received relatively small attention in theoretical meteorological or oceanographical literature, are the ones that really give some information concerning the dynamics of the system.

D. Wake Stream Theory

We shall now consider the balance of forces in a steady, deep current flowing through an ocean basin of uniform depth. The axis of the current coincides with the x-axis. It is assumed that the motion is two-dimensional, horizontal and, because of the depth of the basin, very nearly free from horizontal divergence. Under these conditions and omitting insignificant terms, the equations for the relative motion are

(29) $$\rho \frac{du}{dt} = \rho f v - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}$$

(30) $$\rho \frac{dv}{dt} = -\rho f u - \frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x}.$$

τ_{xy} represents the x-component of the shearing stress acting across a plane normal to the y-axis. τ_{yx} may be interpreted in a similar fashion.

To eliminate the rotation, subtract the deflecting forces and the balancing Coriolian pressure gradients (26), (27) from the forces acting on the system. The result is

(31) $$\rho \frac{du}{dt} = -\frac{\partial p_r}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}$$

(32) $$\rho \frac{dv}{dt} = -\frac{\partial p_r}{\partial y} + \frac{\partial \tau_{yx}}{\partial x},$$

in which equation the "residual" pressure p_r is given by

(33) $$p_r = p - p_c.$$

With the aid of these equations the motion may be analyzed as if the basin were at rest, and the total pressure were given by p_r.

Consider now a current moving under its own momentum and produced by discharging water into the basin through a jet. The theory for such a current system was first developed by Tollmien.[8] The two-dimensional case has been studied experimentally by Förthmann[9] and recently, at the author's suggestion, by Peters and Bicknell.[10] Tollmien's theory for the symmetrical two-dimensional "wake stream" will be outlined below.

With the aid of the equation of continuity the first equation of motion (31) may be transformed and gives

(34) $$\rho \frac{\partial u^2}{\partial x} + \rho \frac{\partial uv}{\partial y} = -\frac{\partial p_r}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}.$$

Assuming that the current has definite boundaries, defined by the condition that u and τ_{xy} both vanish, we may integrate this equation with respect to y and obtain

35) $$\frac{\partial}{\partial x} \int \rho u^2 dy = -\int \frac{\partial p_r}{\partial x} dy.$$

The integration extends across the entire width of the current. Theory and observations show that the term on the right side of this equation is small. Thus, as a first approximation

(36) $$\int \rho u^2 dy = \text{constant},$$

i.e. *the momentum transport through any transversal section is approximately constant.* The experimental verification of this statement will be furnished below.

The approximate constancy of the momentum transport implies that

(37) $$\frac{\partial p_r}{\partial x} = 0.$$

Consequently the equation of motion reduces to the form

(38) $$\rho \frac{\partial u^2}{\partial x} + \rho \frac{\partial uv}{\partial y} = \frac{\partial \tau_{xy}}{\partial y}.$$

In order to integrate this equation, Tollmien assumes that the shearing stress is given by

(39) $$|\tau_{xy}| = \rho l^2 \left(\frac{\partial u}{\partial y}\right)^2,$$

and the mixing length l by

(40) $$l = cx,$$

where c is a constant. One solution is then obtained by assuming that the stream function ψ has the form

(41) $$\psi = \sqrt{x}\, F(\eta) \equiv \sqrt{x}\, F\left(\frac{y}{x}\right),$$

where y is counted from the axis of the symmetric wake stream and increases to the left. Thus

(42a) $$u = \frac{\partial \psi}{\partial y} = \frac{1}{\sqrt{x}} F'$$

(42b) $$v = -\frac{\partial \psi}{\partial x} = -\frac{1}{\sqrt{x}} [\tfrac{1}{2} F - \eta F'].$$

$$\left(F' = \frac{dF}{d\eta}\right)$$

Since the boundaries of the wake stream are defined by the condition that u and τ_{xy}, i.e. $\partial u/\partial y$, vanish, it follows that these boundaries must coincide with the straight lines

(43) $$\eta = \eta_1, \ \eta = \eta_2 = -\eta_1,$$

where y_1 and y_2 are simultaneous roots to the equations

(44) $$F' = 0, \ F'' = 0.$$

It is easily seen that the above expression for u makes the momentum transport constant. If we integrate the equation of motion with respect to y, it follows that

(45) $$\rho \int_{y_1}^{y} \frac{\partial u^2}{\partial x} dy + \rho uv = \tau_{xy},$$

where y_1 refers to the right boundary of the wake stream. Substitution gives

(46) $$FF' = 2c^2 F''^2.$$

A first integral to this equation is obtained without difficulty, but the final solution is best expressed through development in series. One of the two integration constants is determined from the fact that v, and consequently F, must vanish in the axis of the current ($\eta = 0$). The second constant is needed to give the momentum transport its prescribed value and enters as a factor with which the expression for F is multiplied.

The mass transport T through a given section is given by

$$T = \rho \int u dy = \rho \sqrt{x} \left[F(\eta_2) - F(\eta_1) \right]. \tag{47}$$

Thus the mass transport increases downstream while the momentum transport remains constant. This evidently implies that there must be an inflow towards the wake stream from the surrounding fluid. The magnitude of the inflow may be determined by computing the values of v for η_2 and η_1, the boundaries of the current. The width of the current, b, is obtained from

$$b = y_2 - y_1 = x(\eta_2 - \eta_1). \tag{48}$$

Thus the current width increases downstream.

It appears from the expression for the mass transport (47) that this quantity vanishes for $x = 0$. Since all wake stream experiments are made with jets of finite dimensions and finite outflow it follows that the origin for the x-coordinate must be placed a short distance inside the jet.

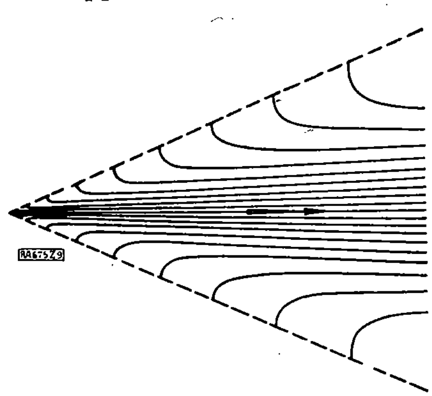

FIG. 1.—Theoretical stream lines in two-dimensional wake stream, according to Tollmien.

Fig. 1 shows the stream lines according to Tollmien and Fig. 2 and Fig. 3 give the theoretical distribution of transversal and axial velocities. Fig. 4 gives a non-dimensional representation of some of the results of Peters' and Bicknell's measurements. Finally in Fig. 5 their observed maximum velocities have been plotted against x. From the two last diagrams it may be inferred that in these measurements the momentum transport was very nearly constant.

The angular spread of the wake stream seems to vary greatly. It depends apparently upon the character of the jet and of the flow as it leaves the jet, but also on conditions in the surrounding fluid. In Peters' and Bicknell's case it varied between 8° and 14°. The angular spread varies in the same sense as the constant c introduced above. There are some indications that c decreases with increasing Reynolds number.

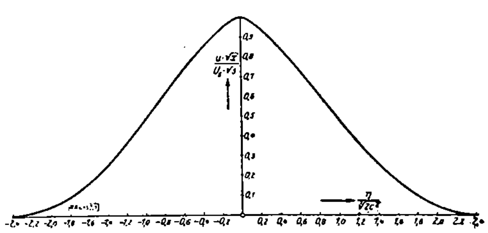

FIG. 2.—Non-dimensional representation of theoretical distribution of axial velocities in two-dimensional wake stream, according to Tollmien.

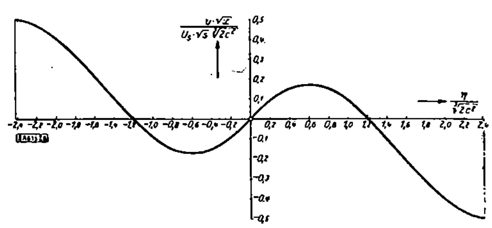

FIG. 3.—Non-dimensional representation of the theoretical distribution of transversal velocities in two-dimensional wake stream, according to Tollmien.

It is of interest to determine the distribution of vorticity within the wake stream. It is given by

(49) $$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

or, after substitution of the expressions for u and v,

(50) $$\zeta = \frac{1}{x\sqrt{x}}\left[\tfrac{1}{4}F - \eta F' - (1+\eta^2)F''\right].$$

Along the boundaries of the wake stream u and $\partial u/\partial y$ vanish. Consequently F' and F'' vanish. Since $v \neq 0$ along the boundaries it follows (from 42b) that $F \neq 0$ along the edges

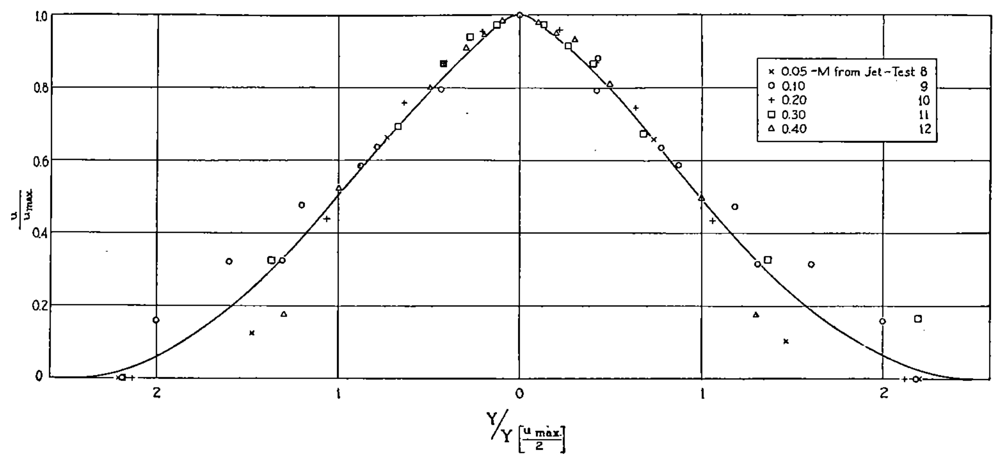

FIG. 4.—Non-dimensional representation of the observed distribution of axial velocities, according to measurements by Peters and Bicknell (the full line represents the theoretical distribution given by Tollmien).

of the wake stream and thus the vorticity does not vanish there. In the absence of frictional forces outside the wake stream the motion of the water drawn in along the edges should be very nearly irrotational. This discrepancy indicates that Tollmien's solution is only approximately correct. It follows from (50) and (42b) that the vorticity at the boundaries is given by

(51) $$\zeta = -\frac{v}{2x},$$

and consequently the theoretically prescribed vorticity decreases rapidly downstream. The prescribed vorticity is cyclonic along the left edge, anticyclonic along the right.

It is now possible to determine the shape of the free surface of the wake stream in the rotating basin. The velocity distribution is, according to the previous reasoning, independent of the rotation of the system. The total pressure gradient is given by

(52) $$\nabla p = \nabla p_c + \nabla p_r,$$

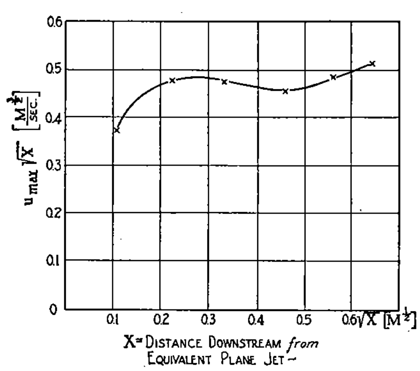

FIG. 5.—The product of axial maximum velocity and the square root of the distance from the jet as a function of the latter distance, according to measurements by Peters and Bicknell.

and since the residual pressure gradient is small it follows that the total pressure gradient very nearly agrees with the gradient of the Coriolian pressure field. If the elevation of the free surface above its equilibrium position be indicated by δ it follows that

$$\nabla p = \rho g \nabla \delta. \tag{53}$$

Thus

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \delta}{\partial x} = \rho f v = -\rho f \frac{\partial \psi}{\partial x} \tag{54}$$

and

$$\frac{\partial p}{\partial y} = \rho g \frac{\partial \delta}{\partial y} = -\rho f u = -\rho f \frac{\partial \psi}{\partial y}. \tag{55}$$

Consequently

$$\delta = -\frac{f}{g} \psi + \text{constant}. \tag{56}$$

Thus stream lines, isobars and lines of equal deformation of the free surface coincide.

Because of the prescribed inflow it follows that the water surrounding the wake stream cannot be at rest. Neglecting frictional forces outside the wake stream and considering the fact that the motion there is two-dimensional and, on account of the depth of the basin, very nearly free from horizontal divergence, it follows that the slow motion of the surrounding water must be very nearly irrotational.

Within the wake stream itself the deviations from geostrophic motion caused by the lateral shearing stresses are very nearly offset by the deviations due to inertia. The principal value of the preceding analysis lies in the establishment of the fact that the volume transport of a given current under the influence of lateral stresses must increase downstream. *Thus the wake stream, which appears to be a divergent current, is actually drawing in water from the surroundings.*

E. Wake Stream in Stratified Medium

The current system described in the preceding section is of limited interest only since it fails to take into consideration the stratification observed in the sea. It is an established fact that all well developed ocean currents are confined mainly to the troposphere. In the underlying stratosphere, which is separated from the troposphere by a transition zone of marked vertical stability, the observed motion is very sluggish. To some extent the effect of this stratification may be taken into account through the assumption that the basin is filled with two homogeneous, incompressible bodies of water; and that the motion, as a result of the stability of the internal boundary, is restricted to the upper layer.

An attempt will now be made to analyze the behaviour of a wake stream in such a basin. In spite of the simplifying assumptions introduced above, the system is too complicated to permit a detailed mathematical discussion and we are forced to restrict ourselves to a qualitative discussion of some of the principal characteristics of the motion.

The x-axis coincides with the axis of the current and points downstream, the y-axis points left. The density of the upper, lighter layer is ρ, that of the resting, lower and heavier layer is ρ'.

In accordance with the results of the preceding discussion it will be assumed that the momentum transport is constant although it will be found later that this assumption must be modified. It follows from (35) that the residual horizontal pressure gradient ∇_{pr} in the upper layer must be negligibly small. *The lines of constant deformation of the sea surface must, therefore, coincide with the isobars and stream lines in the upper layer.*

Let D be the actual thickness of the upper layer and D_0 the thickness of this layer in the undisturbed state in the absence of motion. Let K represent the depth of a certain fixed level in the lower, resting water layer ($K > D_0$). Then the pressure at this level is given by

$$p = \rho g D + \rho' g (K - D). \tag{57}$$

The depth K may be written

$$K = K_0 + \delta, \tag{58}$$

in which the expression K_0 is the depth of the level under consideration in the undisturbed case. Since there is no motion below the boundary, it follows that the horizontal pressure gradient at the level K must vanish. Thus

$$\rho g \nabla D + \rho' g \nabla (K - D) = 0 \tag{59}$$

and consequently

$$\nabla \delta = \frac{\rho' - \rho}{\rho'} \nabla D. \tag{60}$$

Combining (60) and (55) one finds

$$D - D_0 = \frac{\rho'}{\rho' - \rho} \delta = -\frac{f}{g} \frac{\rho'}{\rho' - \rho} \psi + \text{constant}. \tag{61}$$

Thus also the lines of constant depth of the internal boundary coincide with the stream lines.

The flow through a vertical section across the current system is obtained from

$$\rho f u = -\frac{\partial p}{\partial y} = -\rho g \frac{\partial \delta}{\partial y}, \tag{62}$$

or, through substitution of D for δ with the aid of (60),

$$\rho f u = -g \frac{\rho}{\rho'} (\rho' - \rho) \frac{\partial D}{\partial y}. \tag{63}$$

Since u is positive, the internal boundary between the two layers must dip down towards the right. Fig. 6 shows the general form of the internal boundary in a section across the current. In the computation of this diagram it was assumed that the velocity profile could be represented to a sufficient degree of accuracy through an equation of the form

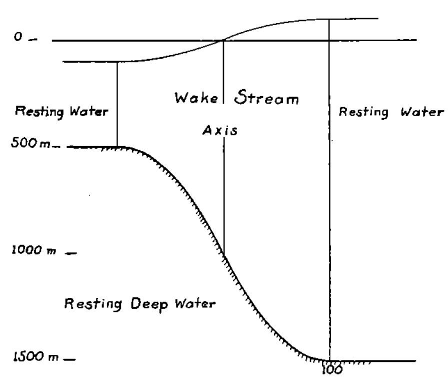

FIG. 6.—Theoretical cross-section through wake stream in stratified ocean under equilibrium conditions.

$$u = u_m \left(1 - \frac{y}{b}\right) \tag{64}$$