GEOSTROPHIC TURBULENCE

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1 INTRODUCTION AND EQUATIONS

Geostrophic turbulence is the chaotic, nonlinear motion of fluids that are near to a state of geostrophic and hydrostatic balance. Thus, 
\[ \mathbf{f} \times \mathbf{u} \approx -\nabla p/\rho - \mathbf{g}, \]
where \( \mathbf{f} \) is twice the component of the planetary rotation vector, \( \Omega \), in the local vertical direction, parallel to \( \mathbf{g} \), which is the acceleration due to gravity, plus the mean centrifugal effect of \( \Omega \). Pressure is denoted by \( p \); \( \mathbf{u} \) is velocity; and \( \rho \) is density. This balance occurs at time scales greater than a day, and length scales greater than tens of kilometers (for the Earth's oceans) or hundreds of kilometers (for the atmosphere). More specifically, we require that the Ekman number (\( E = v/f H^2 \)), the Rossby number (\( Ro = U/f L \)), and the nondimensional frequency \( [(fT)^{-1}] \) each be small. \( U \) is a representative fluid velocity, \( L \) is its horizontal length scale, \( H \) its vertical length scale, \( T \) its time scale, and \( v \) is the kinematic viscosity.

The geostrophic-balance equation is a singular perturbation with respect to neglected accelerations and frictional forces. The curl of the full momentum equation is unaffected by the dominant part of the pressure force and gives a fully predictive potential vorticity equation,

\[ \frac{Dq}{Dt} = \text{forcing} - \text{dissipation} + O(Ro). \quad (1.1) \]

Here, \( q \) is the potential vorticity, defined in several forms below, and \( D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla \) is the Lagrangian acceleration. The derivation of the vorticity equations appropriate to a spherical planet is a lengthy process; here we take them as given. The dominant small parameters are \( Ro \), the fluid depth/planetary radius, the aspect ratio of the motion, \( H/L \), and the ratio of Coriolis to buoyancy forces, \( f/N \) (where \( N^2 = (-g/\rho)\partial p/\partial z \)) [see, for example, Phillips (1963), Miles (1974), and Grimshaw (1975)].

A useful form of (1.1) (not restricted to geostrophic flow) has

\[ q = q_1 = (\nabla \times \mathbf{u} + 2\Omega) \cdot \nabla \rho/\rho. \quad (1.2) \]
where \( \rho \) is the incompressible density (Ertel 1942) and \( q_1 \) is the component of absolute vorticity normal to density surfaces, multiplied by the inverse separation between adjacent surfaces. In this direction, the twisting due to buoyancy, \( \nabla p \times \nabla \rho^{-1} \), has no direct effect. Equation (1.2) is equivalent to Kelvin's circulation theorem written for a material circuit lying on an isopycnal (constant-density) surface.

Scaled for Boussinesq, quasi-geostrophic flow, (1.1) involves a potential vorticity

\[
q = q_2 = \nabla^2 \psi + \frac{\partial}{\partial z} \left[ \frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right] + \beta y - \left[ \delta(z) - \delta(z + H) \right] \frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} + \delta(z + H) f_0 h^*;
\]

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \partial(\psi, \vec{v})/\partial(x,y),
\]

\[
f = f_0 + \beta y, \quad \vec{v} = (-\partial \psi/\partial y, \partial \psi/\partial x), \quad h = H - h^*(x,y),
\]

\[
\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2,
\]

where \( \psi \) is a stream function for the horizontal velocity (nondivergent to order \( R_0 \)). Vertical velocity is dynamically significant, however, and works through the stretching of planetary vorticity (second term). The spherical geometry is approximated by linear dependence of \( f \) on \( y \) (\( x \) and \( y \) are east-west and north-south Cartesian coordinates). The delta functions account for the intersection of isopycnal surfaces with the top and bottom boundaries, which lie at \( z = 0 \) and \( z = -h = -H + h^*(x,y) \).

Thus \( q_2 \) contains flow-dependent terms, the vertical vorticity and vortex stretching, and also environmental effects, such as the shape and mean rotation of the bounding planet.

The continuously stratified fluid can be approximated by a series of constant-density layers, for which (in the \( i \)th layer)

\[
q = q_3 = (\nabla^2 \psi_i + f)/h_i, \quad h_i = \eta_{i+1} - \eta_i + H_i,
\]

\[
p_{i+1} - p_i = g \Delta \rho \eta_i
\]

Here, \( \eta_i \) is the perturbed height of the \( i \)th interface, analogous to \( (-f/N^2) \partial \psi/\partial z \).

Much of the classical theory for large-scale circulations applies simplified forms of (1.1) with specified forcing and laminar friction. The goodness of the theories depends upon the accuracy of the potential vorticity principle and also upon the strength of eddy transports subsumed by (1.1), but neglected in the theory. It is very unlikely that a strongly turbulent fluid can be understood, even in the mean, until the nature of its energy-containing eddies is rationalized.

Conservation of this single scalar potential vorticity puts powerful constraints on the flow. It suggests that knowledge of the deformation
of material surfaces may give information about the dynamics. Conservation of $q$ following the fluid has motivated exact solutions to nonlinear evolution problems; for example, the theory of formation of fronts (Hoskins & Bretherton 1972).

The qualitative properties of geostrophic turbulence that most concern us here are:

1. the ability of chaotic flow to extend contours of constant $q$, hence cascading $q$ to small scale;
2. the inhibition of this cascade by large-scale environmental fields of $q$ [say, given by $Q(x)$];
3. the enhancement of the cascade by small-scale environmental variations in $Q$; and
4. the ability of turbulence to transport potential vorticity systematically down its mean gradient, $VQ$.

Together, these events decide the sense of evolution of the energy-containing eddies, the balance between turbulence and wave motion, the conversion between potential and kinetic energy, the nature of time- or ensemble-mean circulations generated along geostrophic ($Q =$ constant) contours (when the contours are closed), and the mean circulation driven across $Q$ contours (when they are blocked by lateral boundaries).

As one piece of evidence for the presence of mesoscale turbulence, Figure 1 shows a NOAA satellite photograph of the sea-surface temperature in the western North Atlantic. The $O(100 \text{ km})$-wide eddies are of the same size as the Gulf Stream that spawned them, yet are much smaller than either the ocean basin or the broadest components of the wind-driven circulation.

Because of the steeply decreasing spectral energy with increasing wave number, geostrophic turbulence tends, by nature, to be very dependent on the details of the energy-containing eddies, and the forcing effects that created them. A vast range of particular flow problems is relevant to the subject, but we will be constrained here to describe only a few of these.

2 TWO-DIMENSIONAL TURBULENCE

The prototype for geostrophic eddy fields is the motion of uniform-density, incompressible fluid in two dimensions. The original geophysical motivations for this study (that the atmosphere is essentially two-dimensional and weakly dissipative of energy) turn out not to be true, but many of the phenomena of geostrophic turbulence find expression
Figure 1. A sea-surface temperature photograph of the Gulf Stream leaving Cape Hatteras. Darker shades represent warmer water. This front between polar and subtropical water regularly produces geostrophic eddies. Courtesy of NOAA Environmental Satellite Service.
in two-dimensional turbulence. Without this kinship, two-dimensional turbulence would be of less interest, a kind of stochastic complement to 19th century hydrodynamics.

With \( q \equiv \zeta \), the potential vorticity equation is

\[
\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x,y)} = \nu \nabla^4 \psi - R \nabla^2 \psi
\]

(2.1)

\[ \zeta = \nabla^2 \psi, \quad \mathbf{u} = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right), \]

where \( \nu \) is the kinematic viscosity and \( R \) the Ekman drag at the lower boundary, \( R = (v/f H^2)^{1/2} = E^{1/2}f \). Define \( \varepsilon(k) \) \( dk \) to be the kinetic energy, \( \frac{1}{2} | \nabla \psi |^2 \), contained in Fourier components with wave numbers \( |k| \) between \( k \) and \( k + dk \). Then (2.1) yields energy and enstrophy (\( = \text{squared vorticity/2} \)) equations

\[
\frac{\partial}{\partial t} \int \varepsilon \, dk = -2\nu \int k^2 \varepsilon \, dk - 2R \int \varepsilon \, dk < 0,
\]

(2.2)

\[
\frac{\partial}{\partial t} \int k^2 \varepsilon \, dk = -2\nu \int k^4 \varepsilon \, dk - 2R \int k^2 \varepsilon \, dk < 0.
\]

Taylor (1917) established that the many-fold stretching of vortex lines is necessary to the cascade of energy from finite scales to small; with the vortex stretching term \( \zeta \cdot \nabla \mathbf{u} = 0 \) identically, hence the energy dissipation \( \rightarrow 0 \) with \( \nu \). For a finite initial scale of motion, the right sides of (2.2) both vanish for finite time, leaving conservation of total energy and enstrophy (i.e. \( \frac{1}{2} \) squared vorticity), \( \partial / \partial t \int \varepsilon \, dk = \partial / \partial t \int k^2 \varepsilon \, dk = 0 \).

Imagine an initial-value problem in which energy occupies a narrow band of wave numbers. It is most plausible that inertial interactions will cause a spread of energy to adjacent wave numbers. Suppose that this spreading is characterized by growth of the dispersion of \( \varepsilon(k) \) (about its center), i.e.

\[
\frac{\partial}{\partial t} \int (k - k_1)^2 \varepsilon \, dk > 0,
\]

where \( k_n^2 = \int k^n \varepsilon \, dk / \int \varepsilon \, dk \). With the conservation equations, this gives

\[
\frac{\partial k_1^2}{\partial t} < -2(\nu k_1^4 + R k_1^2) < 0.
\]

The first moment of \( \varepsilon \) moves to a smaller wave number, and finite values of \( \nu \) and \( R \) merely hasten the expansion in scale. The overwhelming
control of the enstrophy invariant prevents an energy cascade to the right. Visualize the problem of redistributing a fixed mass (of density $\varepsilon$) lying initially at $k = k^0$, while conserving its moment of inertia about $k = 0$. No more than one fourth of the mass can be moved to $k = 2k^0$ for, to balance, all of the remainder must be shifted back to $k = 0$ [see Batchelor (1953), Fjørtoft (1953), and Merilees & Warn (1975) for further detail].

These events are accompanied by the distortion in physical space of vorticity contours by the shear of larger-scale eddies (Figure 2). If these $q$ contours behave like typical, marked fluid surfaces, then it is

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Figure 2  Contours of constant vorticity in a barotropic, two-dimensional turbulence experiment. The extension of the initially roundish patterns is evident. The model has $128^2$ degrees of freedom.

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1 Two-dimensional turbulence in a real fluid requires stabilization against three-dimensional disturbances. Strong planetary rotation ($f$) will do so, but at the same time transmits frictional influence from horizontal boundaries to the interior fluid. In practice, two-dimensional turbulence often decays in energy more rapidly than three-dimensional turbulence (if $E^{1/2} > \text{Ro}$) (Ibbetson & Tritton 1975).
inescapable that, on average, they should increase in length with time. A proof [that \( \langle (\delta x)^p \rangle \gtrsim (\delta x^0)^p \) for any positive \( p \), where \( \delta x \) is an infinitesimal separation distance between two fluid particles in an isotropic turbulence field and \( \langle \cdots \rangle \) is an ensemble mean] is given by Cocke (1969) and Orszag (1970a). An exponential growth of the length of a \( q \) contour with time is suggested within an inertial range (Batchelor & Townsend 1959).

In an incompressible fluid the stretching of \( q \) contours requires an increase in the average gradient of \( q \), in direct proportion. This measure of the scale of the vorticity field thus decreases [specifically, \( (\partial/\partial t) k_4 > 0 \)]. The enstrophy dissipates in proportion to the squared vorticity gradient; its enhancement by stretching of \( q \) contours is analogous to energy-dissipation enhancement by vortex stretching in three dimensions [Kraichnan (1967), Batchelor (1969), Lilly (1971)]. The opposing energy and enstrophy cascades (with \( k_1 \) decreasing, \( k_4 \) increasing, \( k_0 \) and \( k_2 \) remaining nearly constant) cause streamline (\( \psi \)) patterns to grow in scale while vorticity is strained into thin filaments [visualize the static deformation, \( \psi \), of a membrane loaded with a circular weight, \( q \), and then by an elongated load of the same weight (\( \nabla^2 \psi = q \))]. Figure 3 shows a typical realization of freely-evolving two-dimensional turbulence, using a \( 128^2 \) Fourier-mode numerical code developed by Orszag (e.g. Herring et al 1974).

The greatest weakness in these arguments for the two-dimensional turbulence cascade is that vorticity contours are, in fact, special, and are related to the motion that distorts them. Isotropic velocity fields can be found that will cause the cascade to run backward (for instance, artificially reverse the velocity field in a realization of inviscid two-dimensional turbulence). Cocke's result, that line segments are stretching on average with the turbulence running either backward or forward, suggests that the meticulous unwinding of vorticity contours is indeed unlikely.

Below, I introduce geophysical effects that defeat the enstrophy cascade by organizing the \( q \) contours. With two-dimensional turbulence, however, the back effect of the strained small-scales on the larger, actively straining eddies can be estimated. The sense of the energy exchange is indicated by heuristic arguments familiar to atmospheric studies (e.g. Starr 1953). Small eddies, if distorted by larger eddies in the "obvious" sense of a passive dye trace, develop Reynolds stress patterns tending to feed energy to the larger scale. The transfer from cyclones to the zonal-average winds in the atmosphere is thought to be an example, although we shall see that several other wave number cascades are active in such a fluid.
Thus, as vorticity patterns are teased out by the shearing of larger eddies, their characteristic wave number \( k \) increases, while their vorticity content per octave of wave numbers \( \sim k^3 \delta \) roughly remains nearly constant. Hence the energy content in this same band \( \sim k \delta \) tends to decrease in time. However, the magnitude of this energy transfer becomes small as the disparity in scale increases between strained and straining eddies. The velocity induced back at large scale becomes slight, due to cancellation among closely packed positive and negative vortex elements. For a rough quantitative estimate, let the ratio of energy densities of the two scales be \( v_k^2/v_p^2 \), which is large if wave number \( k \ll p \). The characteristic transfer time is just the inverse strain rate (or circulation time)

![Figure 3](image-url)

**Figure 3** Time-longitude diagram for \( \psi \) showing the evolution of the scale of the eddies along a single latitude in the periodic box. Time increases downward. The Galerkin treatment of the nonlinear terms was developed, for this model, by Orszag (see Herring et al 1974). Geophysical models were developed by Rhines (1977).
of the bigger eddies $[\sim (k v_e)^{-1}]$. During that time the energy transferred $(\sim v_p^2)$ is a small fraction of the energy already present. Kraichnan (1975) discusses these events in more detail.

**Point Vortices and Thermal Equilibrium Solutions**

Methods of equilibrium statistical mechanics have been applied to inviscid two-dimensional turbulence for the idealized case of a fluid with a finite number of degrees of freedom: ensembles of point vortices [e.g. Onsagar (1949), Kraichnan (1975), Pointin & Lundgren (1977)] and artificially truncated sets of Fourier modes [Kraichnan (1967, 1975), Salmon et al (1976)].

Point vortices, $\psi = \sum_i A_i \ln r_i$, where $r_i$ is the radial distance to the center of the $i$th vortex, move about physical space subject to a Hamiltonian,

$$H = - (2\pi)^{-1} \sum_{i>j} A_i A_j \ln (r_{ij}/r_0),$$

where $A_i$ is the circulation about the $i$th vortex, $r_{ij}$ is the distance between $i$th and $j$th vortices, and $r_0$ is an arbitrary constant. The equations of motion are

$$d\dot{\bar{x}}_i/dt = \partial H/\partial \bar{y}_i, \quad d\dot{\bar{y}}_i/dt = - \partial H/\partial \bar{x}_i,$$

where $(x_i, y_i) \equiv (\bar{x}_i, \bar{y}_i)(\rho A_i)^{-1/2}$ are the coordinates of the $i$th vortex. The equilibrium behavior is found [Kraichnan (1975)] to resemble that of a continuous vorticity distribution, truncated at some uppermost wave number. The cascade of energy in two-dimensional turbulence to large scale has been likened to the negative-temperature equilibria of point vortices, which involve clustering of vortices of like sign. But, the actual continuum cascade involves, rather, increasingly fine-grained vorticity and coarse-grained velocity. Straightforward segregation of the vortices by sign cannot occur regularly, because it would involve an increase in kinetic energy. Charney (1960) generalizes point vortices to a stratified fluid.

For continuous vorticity distributions, the inviscid, truncated Fourier system lies on the intersection of a hypersphere ($E = \text{constant}$) and a hyper-ellipse ($k^2 E = \text{constant}$) in a phase space whose coordinates are the respective Fourier amplitudes. Assuming equal probability for each permitted state, we find a micro-canonical (specified $E$ and $k^2 E$) probability distribution for energy and enstrophy. This approximates a Boltzmann distribution provided that the energy is not concentrated in a few modes:

$$P_k(x_k) = \pi^{-1/2}(x + \beta k^2)^{1/2} \exp (-\alpha E_k - \beta Z_k).$$
$E_k, Z_k$ are the energy and enstrophy at discrete wave number $k$. If the lowest few wave numbers are not significantly energetic, the energy spectrum is

$$\varepsilon(k) = \frac{1}{2} \pi k (\alpha + \beta k^2)^{-1}.$$

It is plausible that transient, dissipative cascades proceed in the direction of thermal equilibrium. For initial conditions corresponding to a peak of energy at intermediate $k$, with an increasingly dense set of wave numbers present, $\alpha$ and $\beta$ are both positive. The corresponding equipartition has energy confined to small and moderate $k^2 (\leq \alpha/\beta)$ and enstrophy confined above $\alpha/\beta$. With added geophysical constraints, such equilibria describe the actual flow more closely.

**Evolution of Unforced Turbulence**

A succinct expression of the two-dimensional turbulence cascade is given by Batchelor's (1969) similarity solution. Viscous effects are negligible for finite time, as $v, R \to 0$, so that the only quantities determining the energy spectrum, $\varepsilon(k,t)$, are $U, K, \text{and } t$, where $\frac{1}{2} U^2 = \int_0^k \varepsilon dk$. If the detail of the initial conditions is lost, $\varepsilon$ should adopt a form

$$\varepsilon(k, t) = \frac{1}{2} U^3 t g(Ukt), \quad (2.3)$$

$g$ being an undetermined shape, $\int_0^\infty g(\xi) d\xi = 1$. The spectral peak migrates to small $k$ like $(U/t)^{-1}$. Its magnitude increases like $t$. Energy is conserved, but the total enstrophy, $\int k^2 \varepsilon dk$, decreases rapidly, like $t^{-2}$. The similarity solution thus disagrees with the inviscid constraints, but in a provocative way. The transfer spectra $F$ and $G$, respectively, for energy and enstrophy are defined by

\[
\frac{\partial \varepsilon}{\partial t} = -\frac{\partial F}{\partial k}, \quad \frac{\partial (k^2 \varepsilon)}{\partial t} = -\frac{\partial G}{\partial k^2},
\]

hence

$$F = -\frac{1}{2} \varepsilon k/t, \quad G = -t^{-1} k^3 \varepsilon + 2t^{-1} \int_0^k (k')^2 \varepsilon dk'.$$

The energy flux is everywhere toward small $k$, while enstrophy flows in both directions. Since $k^3 \varepsilon \to 0$ as $k \to \infty$ for finite time, $G \to 2t^{-1} \times \text{total enstrophy}$. The solution thus fails to be uniformly valid, but implies an upward flux of enstrophy, to where transience or dissipation takes hold. The rate at which the cascade occurs defines a constant $T$,

$$\frac{dk^{-1}}{dt} = TU, \quad T = \int_0^\infty \xi g(\xi) d\xi,$$
the "expansion velocity" of the energy-containing scales. From numerical experiments, $T \approx 3.0 \times 10^{-2}$ (Rhines 1975). This suggests that the average eddy doubles in size after a time rather greater than (but proportional to) the inertial time scale ($k_1 U_1$). The corresponding result for the rightward energy cascade of three-dimensional turbulence (Batchelor 1953) is that, empirically, $U^{-2}dU^2/dt \simeq Bk_{-1}U$, where $B$ is of order unity.

**Inertial Ranges**

Kraichnan (1967) proposed that these events could occur in a statistically steady form, imagining a region of constant generation near wave number $k^0$. The required two similarity ranges have distinct roles. Their dimensions require $F \propto k^2 G$, yet for steady conditions both $F$ and $G$ are independent of $k$. Hence, only one can be nonzero; enstrophy flux vanishes in an energy-carrying range and conversely. This is to be understood by imagining the fluxes away from the input wave number as $t \to \infty$. In the limit, the energy, $\varepsilon$, appearing at small $k$ contains vanishing amounts of enstrophy, and conversely at large $k$. If neither viscosity nor the details of the forcing matter, the only free parameters are the energy flux and enstrophy flux. The dimensions require a $k^{-5/3}$ energy range (below $k^0$) and $k^{-3}$ (log-modified, above $k^0$).

Numerical experiments\(^2\) have verified the qualitative nature of the enstrophy and energy fluxes. Batchelor (1969), Lilly (1969, 1971), and Herring et al (1974) give examples. The best of current computers, like the CRAY-1, can advance a $128^2$ model at about 0.1 sec per time step. Unfortunately, even with $128^2$ Fourier modes [Reynolds number, $Re \lesssim 100$, based on enstrophy dissipation, $Re \equiv \frac{1}{2}(U/vk_4)_{1/3}$], the enstrophy dissipation is significant even at the energy-containing wave numbers, rather than occurring only at the largest $k$. The inertial ranges have not been conclusively demonstrated. Nevertheless the big-eddy structure appears to be insensitive to variations in $50 > Re > 100$. Batchelor's solution (2.3) with $\partial \varepsilon / \partial t$ (enstrophy) $= 2t^{-1}x$ (enstrophy) is free of unknown constants, yet the above numerical simulations seem to lose their enstrophy more slowly than in this prediction (even though the viscosity is not very small).

**Closure Theories**

The ambitious goal of a kinetic theory for the second-order moments of the vorticity has been sought, particularly by Kraichnan's direct inter-

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\(^2\) L. F. Richardson (1922) may have put it too strongly, imagining the first numerical simulation of the atmosphere: "In a basement an enthusiast is observing eddies in the liquid lining of a huge spinning bowl, but so far the arithmetic proves the better way."
action approximation and its successors. The sequence of moment
equations, schematically, is
\[
\begin{align*}
\frac{\partial}{\partial t} \langle \zeta \zeta \rangle &= \langle \zeta \zeta \rangle + \langle \zeta \zeta \zeta \rangle, \\
\frac{\partial}{\partial t} \langle \zeta \zeta \zeta \zeta \rangle &= \langle \zeta \zeta \zeta \zeta \rangle + \langle \zeta \zeta \rangle \cdot \langle \zeta \zeta \rangle + \langle \zeta \zeta \zeta \zeta \zeta \rangle,
\end{align*}
\]
where \( \langle \zeta \zeta \rangle \equiv \langle \zeta(\mathbf{k},t)\zeta(\mathbf{k},t') \rangle \). Only the order of the higher moments is
shown. The superscript \( c \) indicates the cumulant terms irreducible in
terms of second-order moments. The theories applied here suppose that
the role of fourth-order cumulants is to limit the growth of triple
moments, giving them a value set by an effective viscosity, \( \langle \zeta \zeta \zeta \zeta \rangle^c = -\mu \langle \zeta \zeta \zeta \rangle \). This leads to a simple evolution equation,
\[
\left( \frac{d}{dt} + \nu \right) \langle \zeta \zeta \rangle = \theta \langle \zeta \zeta \rangle \cdot \langle \zeta \zeta \rangle,
\]
where \( \theta = (\mu + \nu)^{-1} \) is an unspecified array of second moments. \( \theta \) is a
decorrelation time for the higher moments, associated with both strain
rate and persistance time within the eddy field. The theory, particularly
the Galilean-invariant, simplified test-field model (Kraichnan 1971, Leith
& Kraichnan 1972), shows favorable comparison with numerical two-
dimensional turbulence simulations (Herring et al 1974). Yet simpler
theories are being explored, for example the diffusion-like approximation
(Leith 1968, Pouquet et al 1975, Herring 1977) in which the decorrelation
time \( \theta \) takes the familiar form
\[
\theta(k) = \left[ 2 \int_{0}^{k} p^2 \varepsilon(p) \, dp \right]^{-1/2} \quad t \to \infty, \quad k/k_1 \text{ large},
\]
which is just the total rms shear contributed by wave numbers less than
\( k \). The corresponding enstrophy flux, \( G(k) \), is
\[
G = \frac{1}{2} \partial / \partial k \left\{ k^3 \partial / \partial k \left[ k \varepsilon(k) / \theta(k) \right] \right\}.
\]

3 WAVE PROPAGATION AND TURBULENCE

Mean-state potential vorticity fields, \( Q(x,t) \), are provided by the spherical
shape of the planet, by uneven topography of the lower boundary, and
by the density and velocity fields associated with time-averaged currents.
If any of these effects is strong, it may control the shapes of the \( Q \) surfaces.
The enstrophy cascade relies on the stretching of such surfaces and may
thus be imperiled. These mean fields provide both free paths for steady
circulation of the fluid and a restoring effect for transient motions that
cross the geostrophic contours, \( q = \text{constant} \). With mean currents the
distribution of $q$ may provide a positive feedback for the cross-contour motions, leading to spontaneous growth of eddy intensity.

The problem has some features in common with magnetohydrodynamic turbulence (e.g. Moffatt 1967), and with three-dimensional turbulence in a stratified fluid (e.g. Weinstock 1978), or in a rotating fluid (Ibbetson & Tritton 1975). Here it is simplified, however, because there is but a single dynamical field, rather than two or more, and because of the quasi-conservation of $q$ following fluid particles.

The simplest problem is that of random motions of a homogeneous, constant-depth fluid on a rotating sphere or beta-plane (the tangent-plane approximation to a sphere). Scale analysis of the governing equation,

$$\frac{\partial}{\partial t} \nabla^2 \psi + \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)} + \beta \frac{\partial \psi}{\partial x} = 0,$$

suggested that the wave steepness, $\varepsilon \equiv \beta L^2 / U$, determines whether the flow is dominantly linear or nonlinear ($L$ is the horizontal scale of the motion and $U$ the typical transient flow velocity). For $\varepsilon \rightarrow 0$, plane Rossby waves completely describe the evolution of a prescribed initial velocity pattern:

$$\psi = \sum \psi_k e^{i (k \cdot x - \omega t)}; \quad \omega = -\frac{\beta k \cdot \hat{i}}{k \cdot \hat{k}},$$

where $\hat{i}$ is an eastward-pointing unit vector. A recent review is given by Dickinson (1978). The free geostrophic contours are latitude circles, $y = \text{constant}$.

An unforced decay problem is shown in Figure 4 from Williams's (1975, 1978) spherical experiments. At first, energetic, random eddies initiate the two-dimensional turbulence cascade. The measure of nonlinearity, $\varepsilon$, decreases like $(T^2 \beta U t^2)^{-1}$ and, as it passes below unity, radiation of vorticity begins to inhibit the stretching of $q$ contours. Very efficiently, a random field of "steep" Rossby waves appears, which are elongated in the zonal direction.

The spectral arguments for a "red" cascade in two-dimensional turbulence apply here as well; $\beta$ alters the gross conservation laws for enstrophy and energy only when boundaries are present. The rates of cascade are altered by $\beta$, however. When $\varepsilon$ is small, weak-wave triad interactions occur that require both $\omega, k$ resonance and coincidence in physical space. The $\psi$ pattern in Figure 4, therefore, expands only slowly after reaching unit steepness, at wave number $k \approx k_\beta = (\beta / 2U)^{1/2}$. At this same point in the cascade, fluid particles begin to give up their random walks, and oscillate about latitude lines, $y = \text{constant}$. Experi-
mentally, the cascade rate $T$ appears to decrease from $3.0 \times 10^{-2}$ at $\varepsilon \to \infty$ to about $5 \times 10^{-3}$ at $\varepsilon = 1/2$ (Rhines 1975). Still smaller values are anticipated as $\varepsilon$ continues to decrease.

Another geophysical effect that accentuates the contrast in cascade rates at small and large $\varepsilon$ is spatial inhomogeneity. Wave radiation into the quiet regions can reduce the largest values of $\varepsilon$, smoothing out the energy and, even more abruptly, terminating the enstrophy cascade.

![Figure 4](https://example.com/figure4)

*Figure 4* Free evolution of eddies on a rotating sphere ($\beta$ turbulence) (Williams 1978). After 115 days (lower panel), the two-dimensional turbulence has evolved into a zonally elongated set of propagating, steep Rossby waves. The crossover scale, $\pi k_p^{-1} = L_p$ is 3000 km, quite similar to the width of the current bands. Parameters chosen to resemble the Jovian atmosphere ($U = 20$ m sec$^{-1}$; planetary radius $= 0.7 \times 10^5$ km; one day $= 10$ terrestrial hours).
Schematically (the base plane of Figure 8), we imagine a dispersion diagram in \((\omega, k)\) space. Classical waves lie along families of curves, \((3.2)\), which associate large frequency with small wave number. Turbulent states are plotted on the same diagram by endowing homogeneous turbulence of dominant wave number \(k (\approx k_1 \text{ say})\) and energy density \(\frac{1}{2}U^2\) with a frequency \(kU\). The evolution of \(\beta\) turbulence amounts to motion \([\text{at speed } \partial k_1/\partial t \approx (TU^2)^{-1}]\) from an initial state of small eddies, along a constant-energy path \(\omega/k = U\), until the "soft" barrier is reached at the edge of the wave regime.

Williams's (1978) simulations are intended as a simple model of Jupiter's circulation, in response to random forcing. The planet rotates so rapidly (in a ten-hour period) that, even with flow exceeding 100 m sec\(^{-1}\), barotropic motions become wavelike if they are longer in scale than \(\pi k_\beta^{-1} \approx 10^4\) km, that is a planetary wave number of 20. In a companion study for Earth-like conditions, the transition scale \(\approx \pi k_\beta^{-1} \approx 3000\) km (planetary wave number 4). Random midlatitude forcing produces (still in a homogeneous-fluid turbulence model) plausible bands of eddies, easterly zonal-mean winds at low latitude, and westerlies at high latitude (Figure 5). In both examples the observed circulations have meridional scales near \(\pi k_\beta^{-1}\). We return to this work in Section 6. Surface friction (see also Lilly 1972) of terrestrial magnitude can also exert

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Forced eddies on a rotating sphere (\(\beta\) turbulence). The external driving (which exerts no systematic torque) is confined to the latitude band at which eastward (zonal-mean) flow is visible (it is plotted at the right side of the figure). Parameters chosen appropriate to the Earth. A realistic, predominantly zonal flow is combined with rapidly moving eddies.}
\end{figure}
strong control on the statistically steady form of forced two-dimensional turbulence.

In the deep ocean, with flow speeds of \(5 \text{ cm sec}^{-1}\), away from the most intense currents, the wave-turbulence transition occurs at eddy diameters \(~n k^{-1}_b \sim 200 \text{ km}\). Observations suggest that the energy-containing eddies are slightly smaller (more nonlinear).

A remarkable feature of "\(\beta\) turbulence" is the co-existence of wavelike effects with patently turbulent straining fields. Neutrally buoyant "SOFAR" floats (Rossby et al 1975, Freeland et al 1975) drifting passively

![Time-longitude plot of stream function inferred from neutrally buoyant SOFAR float data in the MODE-73 experiment.](image)

*Figure 6* Time-longitude plot of stream function inferred from neutrally buoyant SOFAR float data in the MODE-73 experiment. The central position is 28°N, 69° 40'W, at 1500 m below the surface. Systematic westward phase propagation occurs, at speeds appropriate to linear Rossby waves. Nevertheless, the fluid is undergoing severely turbulent distortions (Figure 7).
Figure 7  Distortion, over a three-month period, of a polygon connecting 5 SOFAR floats in the MODE-73 experiment. We have sketched the smoothest figure that encloses a fixed area, and, no doubt, much small-scale detail is missing. The pseudo-material surface increases its perimeter at a rate consistent with a strong enstrophy cascade. (The eddy circulation time here $\sim \pi$ 100 km/5 km day$^{-1}$ $\sim 60$ days). Dates and positions are indicated.
at 1500 m depth in the MODE experiment exhibit both westward phase propagation characteristic of linear Rossby waves (Figure 6) and intense elongation of marked fluid elements (Figure 7).

Formulation of a statistical theory for a system encompassing both waves and turbulence is a challenging task, for so many distinct physical events are encountered. Holloway & Hendershott (1977) inserted Rossby-wave dispersion into the test-field closure of Kraichnan (1971). The relaxation time, \( \theta_{-k_{pq}} \), for correlations between wave number components \( k, p, q \), now is postulated to be (asymptotically, \( t \to \infty \))

\[
\theta_{-k_{pq}} = \mu_{k_{pq}}(\mu_{k_{pq}}^2 + \omega_{-k_{pq}}^2)^{-1}
\]

\[
\mu_{k_{pq}} = \mu_k + \mu_p + \mu_q, \quad \omega_{-k_{pq}} = \omega_{-k} + \omega_p + \omega_q,
\]

\[
\mu_k = \nu_k + g^2 \sum \theta_{-k_{pq}} b_{k_{pq}} Z_p \big|_{p+q=k}
\]

where \( g \) is a free parameter, \( b = |k \times p|^4/k^3 p^4 q^2 \) and \( Z_p = \langle \zeta_p(t) \zeta_p(t) \rangle \). At low amplitude, \( \theta = \pi \delta(\omega_{-k} + \omega_p + \omega_q) \), which singles out the resonant wave triads. The free evolution predicted by this kinetic theory for \( \beta \) turbulence exhibits the same transformation to slowly cascading anisotropic waves as do the numerical simulations. If any member of an interacting triad lies deeply within the wave regime, the phase-decorrelation time \( \theta \) decreases and energy transfer is inhibited.

4 STRATIFIED GEOSTROPHIC TURBULENCE

Buoyancy and Coriolis forces are typically of equal importance for geostrophic-scale motions. With a stable density stratification, \( \rho(z) \), the fluid is no longer bound together by rotational stiffness. Vertical shear is allowed, providing that the tilting of planetary vorticity, \( f\partial u/\partial z \), is compensated by buoyancy curl, \( -(g/\rho_0)\hat{\mathbf{j}} \times \nabla \rho \): Strong geostrophic control can exist nevertheless. The potential vorticity in the form of Equation (1.3) remains a quasi-conserved scalar. Stratification is typically heavy enough at planetary scales to limit vertical motion; the perturbation pressure is essentially hydrostatic, \( \partial p/\partial z \approx -g\rho'/\rho_0 \).

Taylor's (1917) demonstration for two-dimensional turbulence of the impossibility of an energy cascade to small, dissipative scales holds also for \( \rho \) turbulence. It does so, even though stretching of vortex lines is now a primary effect that transmits stresses vertically and can lead to a many-fold amplification of the relative vorticity.

Charney's (1971) proof follows: Form \( \iiint \psi(1.1) \, dV \) and \( \iiint \Delta \psi(1.1) \, dV \).
These become

\[ \frac{d}{dt} \iiint |\nabla \psi|^2 \, dV = 0, \quad \frac{d}{dt} \iiint |\Delta \psi|^2 \, dV = 0, \]  

(4.1)

where \( dV \) is a volume element and \( S \) is an area element in the top and bottom boundaries, \( z_1 \) and \( z_2 \). \( \Delta \psi \) is the three-dimensional Laplacian and \( z \) has been rescaled by \( f_0/N \), here constant. The added (vertical) dimension provides for the available potential energy \([\infty \left( \partial \psi / \partial z \right)^2] \) and the vertical vortex-stretching contribution to the potential enstrophy. If isotropy holds with respect to horizontal and rescaled vertical directions, Coriolis and buoyancy forces are of the same importance. Assume this to be so, and ignore boundary effects (appropriate to small eddies in the interior fluid). As in Equations 2.2 the spectra yield \( d |k_1|/dt < 0 \) (now a three-dimensional vector), provided that the dispersion of the spectrum about its mean wave number increases with time. Starting with a narrow spectrum at \( k = k^0 \), no more than \( 1/M^2 \) part of the total energy can reach a large wave number, \( k = Mk^0 \). Batchelor’s similarity solution may be transcribed, to describe self-similar free evolution. Ellipsoidal eddies (in the mean, of aspect ratio \( H/L \) in physical space, proportional to \( f_0/N \)) expand in scale while \( q \) itself is drawn into filaments. The energy and potential enstrophy carrying inertial ranges follow as in two-dimensional turbulence. Energy is now doubly protected from dissipation: the three-dimensional cascade is forbidden, and spin-down due to boundary friction is limited by the stratification.

**Energy Conversion and Growth of Barotropy**

If we lift the restrictions of (rescaled) isotropy and allow horizontal boundaries, new phenomena occur: a cascade from large horizontal scale to moderate scale and a flow of energy from vertically sheared, baroclinic to depth-independent, barotropic states.

Imagine an initial condition in which eddies are even broader in scale than \( NH/f \). The potential energy, \( P \), is proportional to vertical distortion of the constant density surfaces, while the kinetic energy, \( K \), is related to their horizontal gradient. Thus \( P/K \sim (f^2/N^2)(\partial \psi / \partial z)^2 / \nabla_2 \psi |^2 \sim (f L/NH)^2 \), which is then large. Nonlinear effects should, we expect, tend to return \( \psi \) to a state of rescaled isotropy where Prandtl’s ratio \( f L/NH \sim 1 \). This causes a conversion of \( P \) to \( K \). A familiar prototype is an exponentially growing wave upon a rectilinear east-west current, sheared in the vertical. It is a linearized baroclinic instability, the model for growth of cyclone disturbances in the atmospheric westerly winds.
To consider these baroclinic processes in more detail, we examine a two-layer idealization of (1.1). The potential vorticity equation is

$$\frac{Dq_i}{Dt} = 0 \quad q_i = \nabla^2 \psi_i + \hat{F}(\psi_j - \psi_i) + \beta y; \quad j = 3 - i,$$

where $i = 1, 2$ refers to upper and lower immiscible layers of fluid of slightly differing density, $\Delta \rho$ and equal depth $H$. $\hat{F} = f_0^2 / g' H$, where $g' = g \Delta \rho / \rho_0$. The interface height is $f_0 (\psi_2 - \psi_1) / g'$. $\hat{F}^{-1/2}$ defines a length.

Figure 8 Schematic representation of freely evolving, nonlinear eddy fields in a flat-bottom ocean (from Rhines 1977). This is a perspective sketch, in which two (frequency, horizontal wave-vector) planes lie normal to the vertical wave number axis. Turbulent states are plotted on this dispersion diagram by ascribing to them a frequency $kU$, where $k$ is the first moment of the energy spectrum and $U$ the turbulent particle velocity. The position of these states relative to the linear dispersion regions (frequency below solid curves) is crucial. Energy-preserving changes of scale occur, within baroclinic eddies, toward the wave number $\hat{F}^{1/2}$ from either side. There, energy passes downward, from baroclinic to barotropic (depth-independent flow) states, followed by a cascade to small wave number. Where energy meets a dispersion region it tends to stagnate.
scale, the Rossby deformation radius, separating strong vertical coupling \((L \gg \hat{F}^{-1/2})\) from weak coupling \((L \ll \hat{F}^{-1/2})\) where the interface is effectively rigid. This scale is of order 50 km in the ocean, 1000 km in the atmosphere (say, planetary wave number 6).

The simplest baroclinic cascade in this model, with initially small eddies \((k_1^2 \hat{F} \gg 1)\), will act like two uncoupled layers of two-dimensional turbulence. The perspective of Figure 8 exhibits baroclinic motion on the upper plane, barotropic motion on the lower. Path c represents such a cascade, which (as in \(\beta\) turbulence, path b) crosses into the wave regime as the wave steepness \(U/\beta L^2 \to 1\).

Path d, however, reaches the stratification-controlled Rossby scale \(\hat{F}^{-1/2}\) before encountering noticeably the \(\beta\) effect. There, the fluid begins to communicate vertically: pressure perturbations build up as \(L\) increases with energy constant; cyclones above begin to uplift the interface, inducing cyclones below. This effect of vortex stretching is a familiar tendency in synoptic meteorology [e.g. Prandtl (1952), p. 386]. Here it acts forcefully, and path d jumps to the barotropic plane, proceeding once again as nearly depth-independent \(\beta\) turbulence. Salmon et al (1976) show that the inviscid, truncated thermal equilibrium theory has a structure consistent with this evolution, strongly barotropic at large scale.

The change of vertical structure is a simple consequence of potential vorticity conservation (Rhines 1977). Take \(\beta = 0\), \(\nu \to 0\), and set \(\psi_2 = 0\) at \(t = 0\). The lower layer evolves such that \(\nabla^2 \psi_2 + \hat{F}(\psi_1 - \psi_2) = -F \psi_1\), following a fluid parcel (\(\psi_1^0\) is the initial value of \(\psi_1\) wherever the deep particle was at \(t = 0\)). Subject only to the unknown behavior of \(\psi_1^0\), the Fourier transforms \(\hat{\psi}_1\) and \(\hat{\psi}_2\) are related forever, \((k_1^2 \hat{F} + 1)\hat{\psi}_2 = \hat{\psi}_1 + \hat{\psi}_1^0\), where \(\hat{\psi}_1^0\) is the Fourier transform of the initial \(\psi_1\) field distorted as if it were a passive tracer by the \(\psi_2\) field. The ratio of potential to kinetic energy, \(P(k)/K(k)\) follows: \(P(k)/K(k) = \gamma^{-1}[1 - 2\delta/(1 + \gamma^2)]\) where \(\gamma = k^2/2\hat{F}\), \(\delta = (1 + \gamma^2)/(1 + \gamma)\).

If the initial eddies were small, relative to the Rossby radius, \(\hat{F}^{-1/2}\), soon \(\hat{\psi}_2\) will far exceed \(\hat{\psi}_1\), and the above relation shows that any energy reaching \(k = \hat{F}^{1/2}\) has \(\hat{\psi}_2/\hat{\psi}_1 = \frac{1}{\gamma}\). Energy reaching larger scales is ever more barotropic. At larger \(k\), the interface rigidity causes \(\psi_2 \ll \psi_1\). This is the only way in which the lower layer, with vanishing \(q_2\), can allow flow to occur. The energy ratio, \(P/K\), peaks at a value 0.2 near \(k = \sqrt{2\hat{F}}\) and falls to zero like \(k^{-2}\) above, and \(k^2\) below; with \(\hat{\psi}_1^0 \ll \hat{\psi}_1\), \(P/K \approx \gamma^{-1}[1 - 2(1 + \gamma)/(1 + (1 + \gamma)^2)]\) (Figure 9).

If, instead, the initial \(\hat{\psi}_1^0\) field had involved larger eddies, these results would not seem to follow: except that we can then appeal to the enstrophy cascade to remove, eventually, \(\hat{\psi}_1^0 \equiv \hat{q}_2^0\) from finite scales. Thus, the \(\hat{q}_2^0\) field has contours that will be continually extended by the \(\psi_2\) turbulence,
and to the extent that this occurs, the above asymptotic results again obtain. (They tend not to obtain at wave numbers containing little energy. Particularly at large $k$, the rightward cascading field of $\psi_1$ is significant enough to exceed $\psi_1$ itself.) With the initial wave number $k^0 \ll \tilde{F}^{1/2}$, the large eddies are heavy with potential energy, $P/K \sim (k_0^2 \tilde{F})^{-1}$.

Yet after the cascade, the largest value of $P/K$ in the field is 0.2, while much of the flow is barotropic. This amounts to a generalized baroclinic instability, followed by growth of barotropy. The details of $\psi(k) = P + K$ are unknown, but eventually, minimizing enstrophy requires that the energy continually move toward $k = 0$, in a nearly perfect state of barotropy, $P/K \to 0$.

The association of declining potential enstrophy with conversion of potential to kinetic energy may be seen by writing $q^2 = k^2 K + 2(\tilde{F} + k^2)P$. With $\langle \cdot \rangle = \int_0^\infty (\cdot) \, dk$,

$$\overline{q^2}/\varepsilon = \left[ k_p^2 + 2(\tilde{F} + k_k^2)P/K \right]/(1 + P/K),$$

where $k_p^2 = k_p^2 P$, $k_k^2 = k_k^2 K$. Fixing $k_p$ and $k_k$, $q^2$ is monotonically increasing with $P/K$.

The initial evolution of these large baroclinic eddies is also visible in the conservation laws for total energy and $q^2$:

$$\frac{\epsilon_t}{\varepsilon} = 0, \quad k^2 \varepsilon_t = -(k^2 + 2\tilde{F})P_t. $$

---

**Figure 9** The ratio of lower-layer to upper-layer stream-function amplitude, as a function of wave number, for two-layer stratified turbulence ($\rho$ turbulence), after the enstrophy cascade has occurred. Also, the ratio of potential to kinetic energy. This plot closely describes the problem with small initial eddies, and approximately describes the energy-containing region of the general problem.
Figure 10a  Free evolution of initially large-scale eddies (relative to the Rossby scale, $F^{-1/2}$). Here, the initial condition is a single baroclinic Rossby wave, plus noise. This is path e of Figure 8. The initial behavior is a linearized instability (an eddy-mean field interaction), yet mutual eddy interactions quickly take over. The end-state has lost most of its potential energy, evolving into a zonally elongated pattern of energetic, fast-moving barotropic Rossby waves. Time, in months, 2000 km by 2000 km re-entrant ocean. CI means contour interval.
Figure 10b: Time-longitude plot of the shallow and deep stream function, $\Psi_1$ and $\Psi_2$, and the temperature field, $-\eta$, along a single latitude line. The evolution from depth-varying, baroclinic flow, to depth-dependent, barotropic flow causes the temperature eddies to vanish. Westward wave propagation coexists with the turbulent cascades. The propagation speed vastly accelerates after the mode transition.
The right-hand side of the enstrophy equation can act to increase the second moment of \( \varepsilon \) if \( P_t < 0 \). If the initial field is large, \( k^2 \ll 2 \tilde{F} \), and then \( k^2 E_t \sim -2 f \tilde{P}_t \), so that this measure, \( k^2 \varepsilon \), of the wave number content increases if \( P \) is converted to \( K \). This is how \( \rho \) turbulence temporarily avoids the "red" cascade of two-dimensional turbulence (which, nevertheless, follows later on), and cascades toward \( k^2 \sim \tilde{F} \).

The cascade from large baroclinic eddies is shown as path \( e \) in Figure 8. In the presence of \( \beta \) it leads to a remarkable increase in both kinetic energy and wave-propagation speed as the field jumps from slow (baroclinic) to fast (barotropic) modes. The streamline patterns from such a free cascade (Figure 10a) show the breakdown beginning as a baroclinic instability (i.e. eddy-mean flow interaction), but quickly turning into a chaotic state of eddy-eddy interaction. The lateral scale at first shrinks to \( \sim \tilde{F}^{-1/2} \), then expands again as \( \beta \) turbulence. Meteorological numerical models (e.g. Simmons & Hoskins 1978, Williams 1978) suggest that the life cycles of midlatitude weather systems involve cyclic travel along path \( e \). Weaker mean flows, which are only slightly unstable, fail to enter into the \( \rho \) turbulence cascade. Instead, they develop into a subtle interplay of a few Fourier components (e.g. Pedlosky 1972). The time-longitude expression of path \( e \) (Figure 10b), shows an abrupt change in the wave propagation, as the turbulence alters both horizontal and vertical structure. Again, waves coexist with a rapid turbulent cascade.

**Closure Models**

The development of a turbulence model that generalizes the notion of baroclinic instability as a cascade process has been initiated by Salmon (1978) and Blumen (1978). The eddy-damped Markovian model of Orszag (1970b, 1974) is adapted by Salmon incorporating both vertical and horizontal shear in the decorrelation time, \( \mu^{-1} \), of the triple moments. The formulation is very like the diffusion approximation of Pouquet et al (1975) for two-dimensional turbulence.

For a statistically steady cascade driven by large-scale stirring, the spectra compare well with direct computer simulations (Figure 11). They show an overwhelming cascade toward barotropy (path \( e \)). Salmon (and the real atmosphere) demonstrate, however, that boundary friction can resist these inertial cascades, if the spin-down time is less than the inertial time scale, \( \sim U \tilde{F}^{1/2} \). Irregular horizontal boundaries can also defeat the cascade.

There are many observational and experimental studies that bear on \( \rho \) turbulence and yet retain their own distinctive nature; for example, the rotating stratified "annulus" experiments (e.g. Hide 1953, Fultz 1959). Geostrophic turbulence, while it contains the universal aspect of stretching...
of $q$ contours, must always be considered with the rather special forcing agents that create it.

An example from studies at Woods Hole Oceanographic Institution (Schmitz 1978) shows currents at two sites in the North Atlantic. The first, Figure 12a, occurs in a region of modest (5 to 15 cm sec$^{-1}$) current near 28°N, 70°W (the MODE-73 experiment). It shows a waxing and waning of vertical coherence, as if nonlinear "capture" of one depth by another occurred, only to be defeated by friction or bottom topographic scattering. The second, Figure 12b, from near the Gulf Stream, represents the discovery of virtually barotropic currents in the deep ocean (see

Figure 11  Comparison of a closure model for steadily forced baroclinic instability (plus turbulent cascade), with numerical simulations (open circles) (Salmon 1978). $u$ is the barotropic energy spectrum, $\mathcal{F}$ the baroclinic kinetic energy, with an adjustable parameter $\lambda$. The tendency to barotropy is strong.

Figure 12a  Two short records from MODE-73 showing horizontal currents measured at two geographical stations (28°09'N, 68°39'W; mooring 482) and (29°35'N, 70°W, mooring 489). The more quickly varying yet weaker currents in the deep water and brief episodes of great vertical coherence are commonplace. Each line is a horizontal current vector, plotted so that north is up.
Figure 12b  Currents observed at a more energetic part of the North Atlantic, just south of the Gulf Stream (37° 30'N, 55° W). Note different scales from Figure 12a. Here, the currents exhibit 50-day periods, roughly, and have penetrated right to the ocean bottom. Strong, depth-independent turbulence is characteristic of the analogous models. (See Schmitz 1978.)

Schmitz 1978). This energetic site on the Sohm Abyssal Plain, has conditions ideal to encourage the enstrophy cascade to follow path e to its end.

5  TURBULENCE ABOVE TOPOGRAPHY

Irregular bottom topography might seem an unpleasant detail to be appended onto the primary cascades represented in Figure 8. It affects the atmosphere, and particularly the ocean, however, at lowest order in
the vorticity balance (e.g. Manabe & Terpstra 1974, Rhines 1969). In a
homogeneous fluid, variations in depth \( h(x) \) and \( \beta \) each act through
gradients in \( f/h \) to provide fixed fields of potential vorticity. Their
respective length scales, however, are very different; \( \beta \), the north-south
gradient of Coriolis frequency, has the scale of the planetary radius,
while \( h \) is distributed over a broad range, with one-dimensional spectrum
typically proportional to \( k^{-2} \).

The relative strengths of topographic and inertial effects are
measured not by \( \delta \) (the rms topographic height \( \div \) the mean depth across
separations \( \sim L \)), but \( \delta/\beta \), a much larger number for geostrophic
flow. The scattering of linear topographic waves (wave-vector triads
comprised of two Fourier components of flow and one of topography)
sends energy and, more strongly, potential enstrophy, to small scale.
The topography is a source of relative enstrophy, \( \frac{1}{2} |\nabla^2 \psi|^2 \), which rises
to such a level that inertial distortions are significant. This is opposite to
the trend of \( \beta \) turbulence, and Figure 4, in which turbulence more usually
evolved into waves.

The governing equation is \( Dq/Dt = 0 \), now with

\[
q = \nabla^2 \psi + h' + \beta y,
\]

where \( h' = \overline{f_0(h-H)/H} \), with \( H = \overline{h} \), the area-average mean depth, and
\( h'/H \ll 1 \). We argued heuristically that the production of relative enstrophy,
\( \frac{1}{2} |\nabla^2 \psi|^2 \), should be positive; conservation of the area-average potential
enstrophy, \( q^2 \) then implies that mean correlation should develop between
vorticity of the flow, and potential vorticity of the mean state,

\[
\partial/\partial t (h' + \beta y)(\nabla^2 \psi) = -\partial/\partial t (\nabla^2 \psi)^2 < 0.
\]

This correlation is the first sign of spontaneously growing time-average
currents: anticyclones above seamounts, and broad westward flow in the
interior of an enclosed basin (for which \( \nabla^2 \psi < 0 \)).

Under the right conditions, it is conceivable that topographic scatter-
ing augments the cascade of \( q \), but not so strongly that the kinetic energy
itself is carried far toward large \( k \). Bretherton & Haidvogel (1976)
suggest that, then, the cascade is efficient enough to bring the fluid to a
state of minimum potential enstrophy, invoking dissipation at the
smallest scales. The time required to remove \( q^2 \) from finite scales is just
\( \sim L/U \).

Temporarily set \( \beta = 0 \) and let \( q_2 = \frac{1}{2} \int \int q^2 \, dS \), \( E = \frac{1}{2} \int \int |\nabla \psi|^2 \, dS \). The
variational problem is to minimize \( q_2 \) subject to fixed energy \( E \). The
boundary conditions are periodicity of \( \psi \) over a large distance \( L_1 \). We have

\[
\delta q_2 + \gamma \delta E = \int \nabla^2 (\nabla^2 \psi + h' - \gamma \psi) \delta \psi \, dS = 0
\]
for all infinitesimal variations, $\delta \psi$, and $\gamma$ is a Lagrange multiplier. The solution requires $\nabla^2 \psi + h' = \gamma \psi$ or

$$\tilde{\psi} = \frac{\tilde{h}}{(\gamma + k^2)}, \quad k^2 \equiv |k|^2,$$

where $\tilde{\psi}$ and $\tilde{h}$ are Fourier transforms. This is a unique, steady solution of the nonlinear equations. The energy level and topographic spectrum determine $\gamma$. At scales small compared with $\gamma^{-1/2}$, $q = \nabla^2 \psi + h' \approx 0$; fluid is carried over small-scale bumps with relative vorticity responding to bottom-slope induced vortex stretching. At large scales ($\gg \gamma^{-1/2}$), $\psi \approx \gamma^{-1} h$; the flow lies directly along geostrophic (depth) contours. $\psi$ is thus a low-pass filtered image of the topographic field (Figure 13).

With $\beta$ reinstated, the analogous problem for an enclosed basin gives simply

$$\psi(x, y) = \sum_k \frac{\tilde{h}}{k^2 + \gamma} e^{ik \cdot x + \frac{\beta}{\gamma} y},$$

plus a term representing inertial boundary layers at the walls. The effect is to add a large-scale westward flow [of the form discovered by Fofonoff (1954)], whose strength $\beta/\gamma$ depends on the initial energy level and topography. The production of a planetary-scale mean circulation is a remarkable consequence of the enstrophy cascade, acting on initially small-scale eddies. The minimum-enstrophy state can only be approximate: an initial spatial distribution of energy that is far removed from this state cannot propagate in space quickly enough to establish it within the decay period for transients (the inertial time scale); the development of true steadiness is controversial; only topographic spectra of "intermediate" slope (say $\sim k^{-2}$) can be expected to allow enstrophy (but not energy) to cascade, and the relevance to a stratified fluid is uncertain.

*Figure 13* The end-state of turbulence over rough bottom topography (Bretherton & Haidvogel 1976): (a) the topography; (b) the final stream function. The initial state for the free decay was a random pattern of eddies uncorrelated with the topography.
Statistical turbulence models of Holloway (1976, 1978), Salmon, Holloway & Hendershott (1976) and Herring (1977) have been applied to "H turbulence," and promise a more complete picture of both the initial adjustment and the end-state.

Salmon et al (1976) give the inviscid equipartition spectrum for H turbulence. The Boltzmann distribution, as in Section 3, applies when there is a rich mix of modes:

\[
\mathcal{P}_i(x_i) = \pi^{-1/2}(\alpha + \beta k_i^2)^{1/2} \exp \left[ -\left( \alpha + \beta k_i^2 \right) (x_i - \langle x_i \rangle)^2 \right],
\]

where \( \langle x_i \rangle = \beta k_i (\alpha + \beta k_i^2)^{-1} Q_i \); \( Q_i \) being the \( i \)th Fourier coefficient of the topographic plus beta effects, \( h'(x,y) + \beta y \). The energy spectrum at equilibrium is

\[
\langle x_i^2 \rangle = \frac{1}{2}(\alpha + \beta k_i^2)^{-1} + \beta^2 k_i^2 (\alpha + \beta k_i^2) Q_i^2
\]

or just the sum of the two-dimensional turbulence equipartition spectrum and the energy of the steady geostrophic contour current, \( \langle x_i \rangle \). This maximum entropy solution has as its ensemble average the minimum potential enstrophy solution.

Holloway (1976, 1978) and Herring (1977) derive closure models for free-decay and steadily forced H turbulence, respectively. Holloway uses a modified test-field model, with a prescription for the decorrelation time, \( \theta \), of the triple moments, which is approximately

\[
(3\theta_{kp})^{-2} \approx \nu_k(k) + \int_0^k Z(p) \, dp + k^{-2} \int_0^k p^2 Q(p) \, dp
\]

\[- \int_0^k (1 - p^2 k^2) R(p) \, dp
\]

for \( k \gg k^0 \). \( R \) is the covariance spectrum of vorticity and topography. The first term is linear viscosity; the second is the strain rate at wave number \( k \) due to larger eddies; the third is the decorrelative effect of topographic Rossby waves [of frequency \( \sim k^{-1} (\int p^2 Q \, dp)^{1/2} \)], and the final term represents the reduction in wavelike restoring forces when flow lies near the geostrophic (\( Q = \) constant) contours. A comparison of the turbulence model with numerical simulations, Figure 14, is encouraging. For a moderately strong initial eddy field, \( \zeta/h' \sim 0.8 \), a significant time-averaged contour flow is found, superimposed on an enduring time-dependent eddy field. In the absence of viscosity the system relaxes to the absolute equilibrium described above.

The energy-transfer spectra in experiments (Rhines 1977) suggest that advective transfer toward small \( k \) (as in two-dimensional turbulence) can rise to balance the topographic scattering to large \( k \), leaving a net spectrum that evolves only very slowly.
Figure 14  Comparison of closure theory and numerical simulations for H turbulence (Holloway 1978). Here, the topographic spectrum, $H$, is $k^2/(1+k^4)$. $Z$ is the vorticity spectrum, $R$ the covariance spectrum between topography and vorticity, and $\bar{Z}$ the predicted static mean circulation. This is a free-decay problem, observed several inertial times after it began.
Herring (1977) has studied the complementary problem of H turbulence driven by stationary random forcing. New topographic-dynamic inertial ranges are predicted. Rather simple forms are given for the transfer spectrum and relaxation time, which accord with the scale analysis of topographic and turbulent strain rates. Increasing the topographic height field has the effect of increasing the static contour flow, increasing the topographic wave frequencies, and decreasing the spectral energy transfer. The topographic parameter, $\delta/\text{Ro}$, determines the ratio of turbulent energy to static-flow energy.

These several approaches concur over the importance of (and form of) the steady-flow mode, as an outcome of a turbulent cascade. It is remarkable to have such explicit predictions of mean-flow generation by turbulence. Yet another view is given in Section 6, below.

Perhaps the weakest aspect of the turbulence models is their insistence on spatial homogeneity. Neither advection nor wave propagation is capable of smoothing the energy of the atmosphere and ocean, which has specific, sparse source regions. Heterogeneity of the bounding orography further aggravates the problem.

When density stratification is added back into the H turbulence problem, the cascade toward a barotropic vertical structure is severely altered. In fact, the baroclinic structure observed to persist in ocean eddies (Figure 12a) may be largely due to bottom topography and severe spatial inhomogeneity (Owens & Bretherton 1977, Rhines 1977).

The H turbulence problem appropriate to the atmosphere is made rather different from these “oceanic” theories by a strong zonal flow, imposed at the outset. Manabe & Terpstra (1974), for example, show that mountains of modest proportions alter the energy transformations of the dominant baroclinic eddies. The advected baroclinic instabilities of a smooth-surfaced planet give way to orographically locked components.

6 MEAN-FLOW INDUCTION

The transport of potential vorticity, if it takes any systematic form, can drive a mean circulation. The geostrophic-contour following flows of Section 5 are examples. For a local view, form the ensemble average of the quasi-geostrophic vorticity equation,

$$\frac{\partial}{\partial T} \langle q \rangle + \langle \mathbf{u} \rangle \cdot \nabla \langle q \rangle = -\nabla \cdot \langle q \mathbf{u}' \rangle + \langle F \rangle$$

(6.1)

where $\mathbf{u}$ and $V$ are horizontal and $q = \langle q \rangle + q'$. Without the eddy contribution, this equation describes the response to forcing (minus dissipation), $F$, in the form of Rossby waves, Sverdrup circulation, or circulations with intrinsically nonlinear mean flow.
The eddy driving term, $\nabla \cdot \langle q'u' \rangle$, can be rewritten in terms of the Lagrangian diffusivity of the fluid, acting on the mean large-scale field $\partial Q/\partial x_i$, with important effects of forcing and dissipation (e.g., Rhines 1978). The Eulerian mean motions that result are the consequence of lateral Reynolds stresses and the vertical transmission of horizontal stress, by a combination of mean Coriolis torque and inviscid wave-drag exerted across deformed isopycnal surfaces.

Let us examine, first, a simple case in which lateral stresses are exerted by geostrophic eddies. Consider a polar $\beta$ plane, with constant depth, density, and Coriolis frequency, $f$, decreasing linearly away from the center. Neglecting forcing and dissipation, we have simple conservation of barotropic potential vorticity,

$$q \equiv \beta y + \zeta,$$

$$\frac{Dq}{Dt} = 0. \tag{6.2}$$

Here $y, v$ are the (inward) radial coordinate and velocity. Integrate over a region within a fixed latitude circle, $\Psi$. The Eulerian, zonally averaged $u$ velocity is then given by

$$\frac{\partial}{\partial t} u_e = \frac{q_v}{v} = -\frac{\partial}{\partial y} w,$$

where $\overline{\int} \equiv \int_\Psi \int dx$ is the integral about the latitude circle. Now a fluid column that would have zero relative vorticity at latitude $y_0$ has potential vorticity $q = \beta y_0 \equiv \beta y - \beta(y - y_0)$. With $v = Dy/Dt, \bar{v} = 0$, and defining $\eta = y - y_0$, it follows that

$$\frac{\partial \bar{u}_e}{\partial t} = -\beta(y - y_0) \frac{D(y - y_0)}{Dt} = -\frac{1}{2} \beta \frac{D}{Dt} \eta^2 \equiv -\beta \bar{K}_{22},$$

where $K_{22}$ is the north-south component of turbulent diffusivity. If the convective part of the right side is small, the above equation becomes

$$\bar{u}_e = -\frac{1}{2} \beta \bar{\eta}^2 \tag{6.3}$$

for an initial state of rest. The neglect of the convective terms leading to Equation (6.3) is not so severe as to require linear wave motion. It implies
\[ u \cdot \nabla \eta^2 \equiv \left( \partial / \partial y \right) v \eta^2 \leq \left( \partial \eta^2 / \partial t \right) \quad \text{or} \quad \gamma(U/c) \left[ (\eta^2)^{1/2} / L \right] \leq 1 \]

where \( \gamma \) is a correlation coefficient between \( v \) and \( \eta^2 \), \( U \) is a scale-particle speed, \( c \) a scale-phase speed, and \( L \) a length scale defining the north-south envelope of variation of \( \eta^2 \). Equation (6.3) is thus valid for either weak disturbances \( (U/c \ll 1) \) or nonlinear fields in which the scale of variation \( L \) of the intensity is much greater than the typical north-south particle displacement.

Random trading of fluid particles across the latitude circle \( \phi \) systematically decreases the net relative vorticity within \( \phi \), yielding an Eulerian mean circulation. This westward momentum appears at "free" latitudes for which Equation (6.2) holds. At forced latitudes, where the wave maker is located, eastward momentum is left behind as a prograde jet whose strength depends upon the nature of the wavemaker: net angular momentum vanishes if the source provides none. In the steadily excited inviscid case the westward circulation at a given latitude builds up to its asymptotic value \( -\frac{1}{2} \beta \eta^2 \) as soon as the disturbance arrives. The eastward jet continues to accelerate indefinitely, to compensate for the presence of westward flow in an ever larger region. The central result, which holds in more general circumstances, is that the depth-averaged flux of potential vorticity equals the force, plus the momentum influx, exerted along \( \phi \) by eddies on the fluid instantaneously occupying the fixed contour, \( \phi \). The association of momentum flux gradient with vorticity flux, and hence with turbulent transport of the (conservative) potential vorticity, was first made by Taylor (1915). Kuo (e.g. 1953), Bretherton (1966), Dickinson (1969), Green (1970), and Rhines (1977) have applied the idea to the atmospheric geometries; it appears that turbulent vorticity transport accounts in part for the maintenance of the mean atmospheric zonal circulation, although nonrandom (monsoon- and orography-related) "eddies" are also important (e.g. Oort & Peixoto 1974).

Experiments demonstrating zonal flows emerging from random forcing on a sphere or \( \beta \)-plane have been performed by Whitehead (1975), de Verdière (1977), and Williams (1978). One experiment, Figure 5, from Williams’s terrestrial simulations shows a clear, large-scale pattern of equatorial easterlies and midlatitude westerlies. de Verdière (1977, Figure 15) has studied the rectified flows in the laboratory, driven by an articulated pattern of fluid sources and sinks.

As a final example, we illustrate the mean-motion patterns found in a model of an enclosed ocean basin driven by steady winds \([\text{Holland (1978)}; \text{see also Robinson et al (1977)} \text{and Sempfner \& Mintz (1977)}]\). Here, a
Figure 15a  The generation of zonal mean circulation on a laboratory “polar $\beta$-plane” (a rotating cylinder of water with a paraboloidal free surface) (de Verdiere 1977). Westward propagating waves are driven by fluid sources and sinks at the periphery; at higher latitudes (nearer the center) a “westward” circulation is driven by the eddies.

Figure 15b  As in Figure 15a, but after halting the forcing of the eddies. The mean circulation remains behind.
new feature is the confinement by meridional boundaries, which block
the geostrophic contours. The interior dynamics of Equation (6.1) now
resembles Sverdrup flow forced by eddies, rather than a much stronger
zonal circulation found when the $Q$ contours are "free."

In the deep water of Holland's model, there is a circulation driven
together by eddies (Figure 16). The plotted particle trajectories illustrate
both the strong eddy mixing and Lagrangian circulation. Whereas the
lateral stresses described above are related to the flux of relative
vorticity, $\langle \mathbf{u}_z \rangle$, the mean motion here is driven by vertical stress trans-

![Graphical representation of particle trajectories](image)

*Figure 16*  Particle trajectories calculated from a numerical model of (Holland 1978). The
domain is a 1000 by 2000 km basin, driven by steady winds (eastward at the central
latitude, westward further north and south). This is the lower layer, which would have no
motion, if it were not for the effect of the upper layer eddies. Both intense dispersion, and
the systematic flow that it creates, may be seen. [Forty particles are released in clusters
of four (from heavy dots).]
mission. This is $f_0 \langle u \eta \rangle / H$, the flux of the isopycnal-layer thickness contribution to $q$ ($\eta$ is the height of the density interface between the two isopycnal layers). Since the flow is quasigeostrophic, $u$ is proportional to the horizontal pressure gradient, and $f_0 \langle u \eta \rangle / H$ is related to the inviscid lateral pressure force $\int \nabla \eta \cdot \mathbf{p} \, dS$ exerted by the upper layer on the lower (e.g. Bretherton 1971, Rhines 1978).

This same inviscid wave drag is operating in the atmosphere. An example is the transmission of westerly stress downward to the ground in the strong midlatitude westerly jets (the depth-integrated eddy-mean flow interaction sees only the lateral Reynolds stresses). Another example is the stratospheric circulation driven by upward radiation of Rossby-gravity waves (e.g. Holton 1975). Although the perturbations are no longer turbulent, the form of the stresses is the same.

7 CONCLUSION

The action of geostrophic turbulence, in cascading potential vorticity, $q$, upward through wave number space, largely determines the development of the energy-containing eddies. It is ironical that increasing the disorder in this way may lead to large-scale flow patterns, vastly unequal partition between potential and (favored) kinetic energy and between baroclinic and (favored) barotropic modes, conversion from turbulence into wave-motion, and from turbulence to a static component of large-scale circulation. It is the turbulent transport of $q$ across physical space (in the presence of mean fields of $q$) that generates time- or ensemble-averaged flow, both eddy-scale circulations about bottom topographic features and planetary-scale drifts.

The spatial and spectral $q$-transports are useful diagnostics for numerical experiments and, when measurable, for observations of nature. The Lagrangian strain rates, seen in the dispersal of marked fluid elements, are a more visible sign of an intense turbulent cascade. The actual rate of disappearance of $q$ from finite length-scales is of great interest, empirically; much of the nature of the mean circulation depends upon the memory of the fluid’s initial potential vorticity.

Energy dissipation is imagined to be a process far slower than $q$-dissipation in these models. Some geophysical examples, however, involve very thick boundary layers that prevent this large-Reynolds-number picture from occurring. Cyclones in the atmosphere, for instance, dissipate their kinetic energy in essentially the inertial time scale. The energy dissipation rate for the ocean is unknown; it hinges on interaction with three-dimensional processes in the fluid interior, rather than boundary turbulence.
Of the effects that resist the enstrophy cascade, we have mentioned wave propagation (due to large-scale $Q$ fields), yet omitted the important possibility that solitons may develop. These permanent-form, isolated solutions of the nonlinear beta-plane equations avoid the cascade by having $q$ constant along streamlines. The various soliton forms [both "round" (Stern 1975, Flierl 1977), and zonally elongated (Larichev & Reznik 1977, Redekopp 1977)] are not far from the states observed in geophysical fluids. They lie near the linear-nonlinear transition, in some cases with the kind of anisotropy observed in the beta-plane experiments. The crucial question is whether many of the boundary-value problems appropriate to geophysical fluids evolve naturally in the direction of these solutions, or instead act out fully the enstrophy cascade.

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Literature Cited

Flierl, G. 1978. Solitons in one- and two-layer fluids. Submitted for publication


Richardson, L. F. 1922. Weather predictions by numerical process. Cambridge Univ. Press
CONTENTS

THE KÁRMÁN YEARS AT GALT, William R. Sears and Mabel R. Sears 1

MODEL EQUATIONS OF NONLINEAR ACOUSTICS, David G. Crighton 11

CAVITATION IN BEARINGS, D. Dowson and C. M. Taylor 35

SELF-SUSTAINED OSCILLATIONS OF IMPINGING FREE SHEAR LAYERS, D. Rockwell and E. Naudascher 67

VORTEX INTERACTIONS, P. G. Saffman and G. R. Baker 95

AIR FLOW AND SOUND GENERATION IN MUSICAL WIND INSTRUMENTS, N. H. Fletcher 123

FINITE AMPLITUDE BAROCLINIC INSTABILITY, John E. Hart 147

SHIP BOUNDARY LAYERS, L. Landweber and V. C. Patel 173

DROP FORMATION IN A CIRCULAR LIQUID JET, D. B. Boggy 207

ROTATING, SELF-GRAVITATING MASSES, N. R. Lebovitz 229

MECHANICS OF ANIMAL JOINTS, Van C. Mow and W. Michael Lai 247

NUMERICAL SOLUTION OF COMPRESSIBLE VISCOUS FLOWS, R. W. MacCormack and H. Lomax 289

WAKES IN STRATIFIED FLUIDS, Jung-Tai Lin and Yih-Ho Pao 317

INTERNAL WAVES IN THE OCEAN, Christopher Garrett and Walter Munk 339

ON THE SPREADING OF LIQUIDS ON SOLID SURFACES: STATIC AND DYNAMIC CONTACT LINES, E. B. Dussan V. 371

GEOSTROPHIC TURBULENCE, Peter B. Rhines 401

THE MEASUREMENT OF TURBULENCE WITH THE LASER-DOPPLER ANEMOMETER, Preben Buchhave, William K. George, Jr., and John L. Lumley 443

SIMILARITY LAWS FOR CONSTANT-PRESSURE AND PRESSURE-GRADIENT TURBULENT WALL FLOWS, A. M. Yaglom 505

INDEXES

AUTHOR INDEX 541

CUMULATIVE INDEX OF CONTRIBUTING AUTHORS, VOLUMES 7-11 551

CUMULATIVE INDEX OF CHAPTER TITLES, VOLUMES 7-11 553