Particle aggregation in a turbulent Keplerian flow

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For the problem of planetary formation one seeks a mechanism to gather small dust particles together into larger solid objects. Here we describe a scenario in which turbulence mediates this process by aggregating particles into anticyclonic regions. If, as our simulations suggest, anticyclonic vortices form as long-lived coherent structures, the process becomes more powerful because such vortices trap particles effectively. Even if the turbulence is decaying, following the upheaval that formed the disk, there is enough time to make the dust distribution quite lumpy.

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I. INTRODUCTION

To rationalize the distribution and the motions of the planets and their satellites in the solar system, Kant\(^1\) and Laplace\(^2\) separately introduced forms of the nebular hypothesis, a conjecture that the solar system formed from a flattened gas cloud or disk. Their prescience has been confirmed: disks do occur in many places in the universe. Since much of the interesting behavior in disks has been ascribed to turbulence, a problem motivated by them is appropriate in a volume dedicated to Robert Kraichnan. That problem is to understand how the planets formed in the primitive solar nebula.

Stars condense from an interstellar medium consisting mostly of gas with an admixture of solid particles called interstellar dust. Both observational and numerical studies suggest that, as the central star contracts, it leaves around it material that contains a good share of the initial angular momentum of the whole system. In this nebula, the centrifugal force balances the stellar gravity in the radial direction and a disk is formed. As the dust settles toward the midplane of this disk it is somehow accumulated into protoplanetary objects.

The general picture has been reviewed by Lissauer\(^3\) who has provided an extensive bibliography. Other summaries exist (such as Ref. 4) and there are a number of introductory expositions (for example, Ref. 5). Two main possibilities have been considered. If the disk is not turbulent, solid dust particles settle into a very thin layer and then can accumulate have been considered. If the disk is not turbulent, solid dust exist

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II. EQUATIONS OF MOTION

To concentrate on vortex dynamics, we restrict ourselves to incompressible flow with

\[ \nabla \cdot v = 0. \]  

We also assume constant viscosity so that the equation of motion is

\[ \frac{Dv}{Dt} = -\frac{1}{\rho} \nabla p - \nabla \Phi + v \Delta v, \]  

where \( \Phi = -\frac{G M}{r} \) is the gravitational potential of the central mass, the protosun.

The key quantity here is the vorticity, for which the evolution equation is

\[ \frac{D\omega}{Dt} = \omega \cdot \nabla v + v \Delta \omega. \]  

In a disk, the large scale flow is nearly two-dimensional and we adopt this simplification here. Then we may express \( v \) in terms of a stream function \( \psi \) such that, in Cartesian coordinates, the velocity field is given by

\[ \begin{pmatrix} \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x} \end{pmatrix}. \]  

The vorticity then has only one component, \( \omega \), and it is related to the stream function by

\[ \omega = \nabla^2 \psi. \]  

Hence Eq. (3) may be written as

\[ \frac{\partial \omega}{\partial t} + J(\psi, \omega) = \nu \omega, \]  

where \( J(\psi, \omega) = \partial_x \omega \partial_y \psi - \partial_y \omega \partial_x \psi \).

In the absence of dissipation \( (\nu = 0) \), Eq. (6) admits an infinite number of conserved quantities, two of which are quadratic invariants; these are the kinetic energy \( E = 1/2 \int (\nabla \psi)^2 dx \, dy \) and the enstrophy \( Z = 1/2 \int (\nabla^2 \psi)^2 dx \, dy \). The conservation of these quantities in the inviscid limit is what makes two-dimensional turbulence so different from three-dimensional turbulence at large Reynolds numbers.\(^{23,24}\) In two-dimensional turbulence there is a direct cascade of enstrophy from large to small scales and an inverse cascade from small to large scales of kinetic energy.\(^{25-27}\)

For freely decaying, barotropic turbulence in shear-free environments, intense long-lived vorticity concentrations are observed to form after an energy dissipation time. These coherent vortices are characterized by a broad distribution of size and circulation and they contain most of the energy and the enstrophy of the system.\(^{28,29}\) The first question we take up here is whether the strong Keplerian shears of accretion disks prevent this well-documented behavior.

III. ASTROPHYSICAL DISKS

The purpose of this section is to provide a general idea about the background for the fluid processes that we study in the next sections. This is called for in a controversial subject such as planet formation since many readers of this journal may not have previously encountered some of the issues raised. On the other hand, the occurrence of disks of many sizes throughout the universe is so well documented that our discussion can safely start from the assumption that one of these once surrounded the protosun. We shall not go into the processes by which disks are thought to form but consider only the simplest case of a disk that is axisymmetric in the large and symmetric about the midplane \( z = 0 \). In the radial direction, the principal support against the gravitational pull of the protosun is provided by centrifugal force, while in the \( z \) direction the support is by a pressure gradient. If the ratio of these two support forces is large, the disk is quite thin, as we shall presume here.

Let the rotational speed at distance \( r \) from the central protosun be \( v \). The centrifugal balance condition is

\[ v^2 = 2 G M/r, \]  

where \( M \) is the mass of the protosun. This describes a Keplerian flow that is stable to small perturbations since it has no inflection points and the Rayleigh discriminant,

\[ \Delta = r^{-2} d(r^2 \Omega)^2 / dr, \]  

is positive. In astronomy, where this stability condition is often met, it is customary to introduce the epicyclic frequency \( \kappa = \sqrt{\Delta} \), whose role in the dynamics is not wholly different from the buoyancy frequency of stably stratified fluids. Since the disk is thin, we may identify \( r \) with the radius in cylindrical coordinates. In our two-dimensional simulations, we retain this meaning of \( r \) and the form \(-G M/r^2\) for the gravitational acceleration.

If the flow remains laminar, there will be a slow viscous transport of angular momentum outward that allows a weak inward spiraling of the material onto the protosun. The two-dimensional version of such motion has been studied in the context of drag reduction theory\(^{30}\) where it is thought that a slow inflow can inhibit boundary layer separation and reduce the drag on airfoils. This process is too slow for astrophysical purposes when it is laminar, but its turbulent analogue has been invoked for the case of astrophysical disks,\(^{31}\) which are called accretion disks because of this inflow. In the turbulent case, it is suggested that the dissipation rate could be high enough to render some disks quite bright. But can we assume that disks are turbulent?

Until recently, many took it for granted that Keplerian disks were turbulent because the Reynolds number in a disk is literally astronomical. On account of (7) and the relatively small viscosity, it was often assumed\(^{32}\) that “The successive rings of gas in the medium will therefore have motions relative to one another, and turbulence will ensue.” But simulations by Balbus and collaborators,\(^{33}\) among others, have not shown nonlinear instabilities in Keplerian disks at the respectable Reynolds numbers they achieved.

Apparent, the circular Keplerian flow is different from plane Couette flow, where nonlinear instability is known to occur, and indeed does so in the calculations run by Balbus et al. Nevertheless, their conclusion is not universally accepted\(^{34}\) and there is an issue of principle to be decided here. One technique that may be decisive is the method of
turbulence and it permits us to study the effect of strong shear in this problem, which turns out to be that anticyclonic vortices are heavily favored.

A. Technical details

The basic quantity that we compute is the vorticity, which we split into two parts: a specified (regularized) Keplerian flow and a disturbance to the basic flow. Thus we write

\[ \omega = \omega_K(r) + \zeta(r,t), \]

where \( \zeta \) is the dynamical quantity of interest.

The imposed background Keplerian velocity profile is, in regularized form, taken to be

\[ v_K(r) = \frac{K}{\sqrt{r}} \left[ 1 - \exp \left( -\frac{r}{r_0}^{3/2} \right) \right], \]

where \( K \) and \( r_0 \) are adjustable parameters. For \( r \gg r_0 \), a conventional Keplerian profile \( K/\sqrt{r} \) is recovered, while for \( r \ll r_0 \), the velocity field reduces to a solid rotation \( Kr/r_0^{3/2} \).

The behavior at small \( r \) is introduced to avoid the singularity at the origin associated with a point mass. This profile has a vorticity field, \( \omega_K = (1/r)(\partial \partial r)(rv_K) \), given by

\[ \omega_K(r) = \frac{K}{2 r^{3/2}} \left[ 1 - \exp \left( -\frac{r}{r_0}^{3/2} \right) \right] \]

\[ + 3 \left( \frac{r}{r_0} \right)^{3/2} \exp \left( -\frac{r}{r_0}^{3/2} \right) \].

For \( r \ll r_0 \), \( \omega_K \approx 2Kr/r_0^{3/2} \) and for \( r \gg r_0 \), \( \omega_K \approx K/2r^{3/2} \).

Our simulations are based on a standard pseudospectral code with 2/3 dealiasing and resolution 512 × 512 grid points in a square box with periodic boundary conditions \( [-\pi, \pi] \times [-\pi, \pi] \). In the simulations discussed below, we have used the value \( r_0 = 0.123 \), which corresponds to ten grid points in the region of rigid rotation. We also adopt \( K = 2.07 \), which gives the mean kinetic energy of the Keplerian shear the value \( E_K = 0.5 \). With these choices, we ensure a Keplerian profile with \( \omega_K \approx 1/r^{3/2} \) for the underlying vorticity and velocity fields in the range \( 0.3 < r < 2 \) (see Fig. 1). For \( r < 0.3 \), the disk rotates rigidly and for \( r > 2 \) the pure Keplerian profile is altered by the periodicity of the domain and the interaction with disk images from the periodic boundary conditions. At \( r = \pi/6 \), the angular velocity is \( 2\omega_K = 5.46 \), which gives a typical rotation time \( T_K \approx 1.15 \). The simulations have been run up to a total time \( T = 20 \).

When we introduce the decomposition (9) into (6), we obtain an evolution equation for the disturbance vorticity \( \zeta \):

\[ \frac{\partial \zeta}{\partial t} + J(\varphi, \zeta) = \nu \Delta \zeta + F, \]

\[ F = -J(\psi_K, \zeta) - J(\varphi, \omega_K), \]

where \( \varphi \) and \( \psi_K \) are the stream functions associated with \( \zeta \) and \( \omega_K \), respectively. The quantity \( F \) represents the effect on the disturbance flow of the imposition of a background
Keplerian shear, which satisfies $J(\psi_K, \omega_K) = 0$. We have assumed that the dissipation acts only on the disturbance field $\zeta$ and we have taken $\nu = 5 \times 10^{-5}$.

As initial condition, $\zeta(r, 0)$, we select a narrow-band random vorticity field with energy spectrum

$$E(k) = E_0 [(m/n)k_0 + k]^{m+n},$$

We have taken $k_0 = 10$, $m = 30$, and $n = 5$. The value of $E_0$ is chosen to control the relative energies of the Keplerian and of the disturbance flow, as specified below. The Fourier phases are randomly distributed initially between 0 and $2\pi$. In order to avoid unrealistic periodicity effects, the disturbance has been limited to a region of radius $r_{\text{max}} = 2$, so that no perturbations occur near the edges of the simulation box.

### B. Formation of anticyclonic vortices

In our simulations, when the initial energy of a disturbance is less than about $10^{-3}$ of the energy in the Keplerian flow, the vorticity fluctuations are sheared away and the disk returns to its unperturbed velocity profile. This behavior is consistent with the linear stability of Keplerian disks. When we begin a calculation with larger perturbation energies than this, an initially random vorticity field forms itself into well-defined vortices, much as in flows with little or no shear, except that our vortices are all anticyclonic. Cyclonic vortices do not form and, even if they are expressly introduced initially, the shear quickly destroys them.

Figure 2 shows the vorticity field for a simulation in which the initial disturbance energy was chosen to be $4 \times 10^{-3} E_K$. The vorticity is shown at times $t = 0$ [Fig. 2(a)], $t = 8$ [Fig. 2(b)], and $t = 15$ [Fig. 2(c)]. The presence of anticyclonic vortices is clearly visible. Because they last many of their own turnover times, such vortices are an example of coherent structures.

The initial sizes of the developing anticyclonic vortices are small, but they merge and grow until they apparently reach a limiting size. The limiting size of the vortices increases with the distance to the center of the disk roughly like $r^{1.5}$ out to $r \sim 1.5$, although small vortices penetrate into the central regions of the disk. In this system, formation and persistence of coherent vortices is possible only when the nonlinear term $J(\psi, \zeta)$ in Eq. (12) is larger than dissipation $\nu \Delta \zeta$, large-scale shear $J(\psi_K, \zeta)$ and differential rotation $J(\psi, \omega_K)$. For Keplerian disks, the range of scales where vortices can persist is thus limited from below by dissipation and from above by shear and differential rotation. Whenever the two limiting scales become too close, vortices cannot survive.

For thinking of the solar nebula, we identify a length scale such that the border of the disk, at $r_{\text{max}} = 2$, corresponds to 7 AU. (An astronomical unit is the mean distance of the earth from the sun.) With this convention, the biggest vortices of our simulation are located at typically 5 AU, at the distance of Jupiter from the center, while the smallest vortices lie at roughly the earth’s distance from the sun.

Once the length scale has been chosen, the time scale is determined by Kepler’s law of rotation. First we recall that the rotation period is $T_K = 1.15$ at $r = \pi/6$ in non-dimensional units. In physical units, this distance corresponds to about 1.83 AU. At this distance, the (physical) revolution time is $T_K^* \sim 2.48 \text{ year}$. Physical time is therefore related to non-dimensional time by

$$t^* (\text{year}) = 2.16 t.$$  

To give the value of the viscosity coefficient a physical meaning, we recall that turbulent viscosity in disks is usually expressed in terms of the dimensionless $\alpha$ parameter by

$$\nu = a H^2 \omega_K^*,$$

where $H$ is the thickness of the disk and $\omega_K^*$ is the Keplerian rotation in the region of interest (in physical units). In the standard model of the solar nebula, $\alpha$ is in the range $10^{-3} - 10^{-2}$ in the region of Jupiter, assuming $H \sim 0.3$ AU. This determines a physical eddy viscosity $\nu \sim 5 \times 10^{-5} - 5 \times 10^{-4} (\text{AU})^2/\text{year}$. In non-dimensional units, it corresponds to $8 \times 10^{-6} - 8 \times 10^{-5}$. The viscosity used in the previous simulations, $\nu = 5 \times 10^{-5}$, lies in this range, indicating that formation of coherent vortices is indeed possible when the parameters take the values commonly estimated for the solar nebula.

### V. THE AGGREGATION PROCESS

Anticyclonic vortices trap small particles. In this section, we study the effectiveness of this process when an explicit solution of the equations of motion is used to provide the vortices. In particular, we ascertain that the slowly decaying vortices in Keplerian turbulence have time to produce significant particle aggregation and so initiate planetary formation before they die away.
A. The particle equation

We next follow the Langrangian motion of dust particles in the flow computed in the previous section. This motion is described in a rigidly rotating frame with an angular velocity that is approximately the mean of the Keplerian rotation over the disk, namely $\Omega = 0.617$.

The study of particle motion in a fluid flow is surprisingly complicated, especially for fluffy, extended flakes. But here, we consider the simplest case of dust particles whose sizes are less than the mean free path in the ambient gas and whose typical density $\rho_d$ is much greater than $\rho_g$, the density of the gas around it. For this case, the equation of motion of a dust particle with position $\mathbf{r}$ is

$$\frac{d^2 \mathbf{r}}{dt^2} = -2\Omega \frac{d\mathbf{r}}{dt} \gamma \left( \frac{d\mathbf{r}}{dt} - \mathbf{u}(\mathbf{r},t) \right) + \left( \Omega^2 - \frac{GM}{r^3} \right) \mathbf{r} \tag{17}$$

where $\mathbf{u}(\mathbf{r},t)$ is the fluid velocity in the rotating frame and $\gamma$ is a friction coefficient. In addition to the gravitational force on the particle, this equation contains the Coriolis force, the centrifugal force, and a phenomenological drag force produced by the ambient medium.

For the rarefied conditions of the protoplanetary nebula, $\gamma \approx \rho_g/a$, where $a$ is the particle radius; this corresponds to the so-called Epstein regime. Our simulations of the particle motion show that the value of $\gamma$ is important in the aggregation process. For large $\gamma$, the particles are rather like tracers and for small $\gamma$ the concentration process is more rapid.

B. Trapped particles

We start our simulation of the Lagrangian trajectories with a uniform distribution of particles moving initially at the local fluid velocity. This leads to the results shown in

FIG. 2. (a) Initial vorticity field of the perturbed two-dimensional Keplerian disk ($t=0$); (b) Vorticity field at $t=8$; and (c) Vorticity field at $t=15$. 

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Fig. 3 for $\gamma=10$. The successive panels display the particle distributions at times $t=2, 8, 15,$ and $20$. To bring out the effect of vorticity, $\zeta$, in Fig. 4 we provide a plot of the time evolution of the average density of particles in three different regions, namely where $\zeta<\zeta_0$ (full line), $-\zeta_0<\zeta<\zeta_0$ (dotted line), and $\zeta>\zeta_0$ (dashed line) with $\zeta_0$ as an estimate of the rms initial vorticity. The plotted densities are divided by the mean density, so that the initial distribution is everywhere unity. We observe that the particles are expelled from the cyclonic regions and concentrated inside the anticyclonic vortices. Simulations with other values of the drag parameter $\gamma$ have confirmed this general picture; the particle concentration process is even more rapid for larger particles associated with smaller values of $\gamma$. A full study of the role of the various parameters appearing in Eq. (17) is reported in Ref. 14 for a kinematic flow field.

These results are qualitatively consistent with earlier empirical work\textsuperscript{12,9} and with theoretical studies in which the particle paths were followed in an \textit{a priori} prescribed flow containing vortices.\textsuperscript{13,14,45} The quantitative result added here is that decaying vortices in a real flow derived from the fluid equations have adequate time to aggregate particles.

Thus, by marching the particle motion along with the simulations that produce the vortices we have been able to see how turbulent fluctuations initially present in the disk draw particles into regions of negative vorticity. Already in
the early evolution of the solar nebula we observe the development of pockets of dust which fill the disk inhomogeneously even as the vortices form. This separation of the dust into different regions is caused by the cyclone/anticyclone asymmetry, and it is so rapid that at $t=2$ (a matter of years) the density of the dust particles in the region $\zeta < -\zeta_0$ has already been increased by a factor of 2. When the vortices do form, they are already surrounded by a large number of dust particles and so the rate of capture is enhanced.

VI. DISCUSSION

Rotating cosmic bodies whose atmospheres are turbulent and can be seen with good resolution display coherent structures—vortices in the atmospheres of planets and magnetic flux tubes in the solar atmosphere. However, many accretion disk theorists feel that this observation is not a good indicator of what happens on disks because, they argue, the strong Keplerian shears in accretion disks will destroy vortices. There is practically no direct observational evidence on this point, but indirect evidence does suggest that coherent structures form in the disks around quasars.46 Our fluid dynamical simulations (see also Ref. 18) show that Keplerian shears feed anticyclonic vortices and destroy only the cyclonic ones. For vortices to form, however, strong perturbations such as might be provided by turbulence are needed.

It is believed that the progenitor of the solar system was so cool that it was not ionized and so magnetic flux would not have been frozen in. Then the powerful magnetic instabilities that render hot disks turbulent would not operate and there is no well established mechanism for sustaining turbulence. Nevertheless, we have argued here that one may call upon a residual turbulence from the formation throes of the solar nebula. The vortices that would have been formed in those initial stages do seem to have time to act significantly on the particle distribution before decaying away. It is even possible that the process might have involved some of the primordial fields in the collapsing interstellar matter. Such details aside, it does seem reasonable to conclude that, once there is some turbulence, anticyclonic vortices form. This in itself is an interesting fluid dynamical process that is likely to take place on conventional accretion disks. The version of the problem that we have looked at has features in common with two-dimensional turbulence, which Kraichnan has done much to explicate.24,27

Of course, the situation is more complicated in the real astrophysical contexts than the version we have presented here and a more realistic version of the theory might be based on shallow layer theory.18 Even if the fluid is compressible, as it is in astrophysical disks, the thin layer approximation is very close to the standard shallow water equations,47 which also give rise to vortex formation if strong disturbances are introduced.48,49 Whether vortices form in disks or not, large vortical disturbances can by themselves redistribute particles. This may in itself be enough to initiate planet formation, but a complete theory would require a more detailed knowledge of how the nebula formed.

In more general terms, the (perhaps) surprising ability of turbulence to render an initially homogeneous distribution of ambient particles very inhomogeneous is likely to have consequences in several fields and deserves further study in the wake of the recent theoretical efforts in this direction.12,45,15 This problem might also profit from a statistical treatment analogous to what has been done in the problem of the advection of a passive scalar, another subject to which Kraichnan has made a significant contribution.

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