Two-Particle Dispersion in Isotropic Turbulent Flows

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Abstract
Two-particle dispersion is of central importance to a wide range of natural and industrial applications. It has been an active area of research since Richardson’s (1926) seminal paper. This review emphasizes recent results from experiments, high-end direct numerical simulations, and modern theoretical discussions. Our approach is complementary to Sawford’s (2001), whose review focused primarily on stochastic models of pair dispersion. We begin by reviewing the theoretical foundations of relative dispersion, followed by experimental and numerical findings for the dissipation subrange and inertial subrange. We discuss the findings in the context of the relevant theory for each regime. We conclude by providing a critical analysis of our current understanding and by suggesting paths toward further progress that take full advantage of exciting developments in modern experimental methods and peta-scale supercomputing.
There is no royal road to science, and only those who do not dread the fatiguing climb of its steep paths have a chance of gaining its luminous summits.

Karl Marx, Das Kapital

1. INTRODUCTION

Arguably the most elemental understanding one can have of any flow field is how particles are moved by the flow. Conceptually, it is far simpler to consider the trajectory of a particle than it is to fully comprehend the velocity vector field. The motion of particles is also important because of its connection with the processes of transport and mixing that impact natural and engineering flows in such profound ways. Indeed, it is the latter application that drew the attention of some of the greatest minds in fluid mechanics to the study of particle motion in turbulence. Taylor’s (1922) original work on single-particle dispersion gave birth to many of the modern statistical tools we use to study turbulence.

But it is Richardson’s (1926) work that forms the central theme of this review. In his seminal study, he examined the relative motion of two particles embedded in isotropic turbulence, establishing the foundations of two-particle dispersion. There is a fundamental link between the formal analysis of pair dispersion and practical problems such as the growth relative to the center of mass of a cloud of contaminants in the atmosphere, nutrients in the ocean, or chemical species in a turbulent reactor. In all these examples that involve fluid flow, a nondimensional parameter that represents the ratio of inertial and viscous forces is the Reynolds number, here defined in terms of the Taylor microscale as

$$\lambda \equiv \sqrt{\frac{15\nu}{\langle u'^2 \rangle \langle \epsilon \rangle}}$$

where \(\langle u'^2 \rangle \) is the root mean square of the fluctuating velocity, \(\nu\) is the fluid kinematic viscosity, \(\epsilon\) is the turbulent energy dissipation rate. Laboratory and industrial flows are characterized by \(\lambda \approx O(10^2–10^3)\), whereas geophysical flows can reach \(O(10^4)\) and higher.

Restricting our attention to particles that are small enough and neutrally buoyant so as to be considered passive tracers, the kinematic equations governing the separation vector \(\mathbf{r}(t)\) in differential and integral form are

$$\frac{d\mathbf{r}}{dt} = \mathbf{w}(t), \quad \mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{w}(t') dt'.$$

where \(\mathbf{w}(t)\) is the relative velocity between the two particles, and \(\mathbf{r}_0\) is the initial separation vector, corresponding to \(\mathbf{r}(0)\). For the case of a turbulent motion, we are interested mainly in the average growth rate of the separation distance. Defining \(\langle r^2(t) \rangle = \langle \mathbf{r}(t) \cdot \mathbf{r}(t) \rangle\) as the mean square separation distance (where \(\langle \cdot \rangle\) indicates an ensemble average), we can derive an exact differential equation for this quantity:

$$\frac{1}{2} \frac{d\langle r^2 \rangle}{dt} = \langle \mathbf{r}(t) \cdot \mathbf{w}(t) \rangle$$

$$= \langle \mathbf{r}_0 \cdot \mathbf{w}(t) \rangle + \int_0^t \langle \mathbf{w}(t') \cdot \mathbf{w}(t') \rangle dt'.$$

In integral form, this becomes

$$\frac{1}{2} \langle |\mathbf{r}(t) - \mathbf{r}_0|^2 \rangle = \int_0^t \int_0^{t'} \langle \mathbf{w}(t') \cdot \mathbf{w}(t'') \rangle dt'dt''.$$

Equations 2 and 3 are nonlinear owing to the implicit dependence of the relative velocity on the separation vector. Indeed, the principal theoretical challenge is to model this relationship.

Relative dispersion goes both ways. One can also ask, given two particles that are separated by a distance \(r(t)\) at time \(t\), on average how far apart were those particles at time \(t - \tau\)? This is called

$$\frac{1}{2} \langle |\mathbf{r}(t) - \mathbf{r}_0|^2 \rangle = \int_0^t \int_0^{t'} \langle \mathbf{w}(t') \cdot \mathbf{w}(t'') \rangle dt'dt''.$$

We can divide the process of dispersion into three distinct regimes based on the separation of the particles relative to the turbulent scales: (a) The dissipation subrange corresponds to $r(t) \ll \eta$, where $\eta \equiv (v^3/\langle \epsilon \rangle)^{1/4}$ is the Kolmogorov length scale; (b) the inertial subrange corresponds to $\eta \ll r(t) \ll L$, where $L$ is the integral length scale; and (c) the diffusion subrange corresponds to $r(t) \gg L$. The analysis and scaling for each subrange are unique, so we discuss each separately below. The diffusion range is statistically equivalent to the long-time limit of the problem of single-particle dispersion initially addressed by Taylor (1922), a topic that is outside the scope of this review.

We limit this review to the forward dispersion of particle pairs in homogeneous isotropic turbulence (see Toschi & Bodenschatz 2009 for Lagrangian statistics of more than two particles). Our goal is to provide a complementary perspective to Sawford’s (2001) review, which focused primarily on modeling approaches. Here we emphasize the empirical evidence obtained from laboratory measurements, field measurements, and direct numerical simulations (DNSs), but whenever appropriate we introduce theory to help interpret the data or place observations into the proper context.

In Section 2 we give the analysis for the dissipation subrange, followed by our discussion of the inertial subrange in Section 3. We offer a discussion of the results in Section 4 that includes a tentative assessment of the theory. The discussion focuses on the more controversial findings in the inertial subrange. Concluding remarks are given in Section 5, including suggestions for future laboratory and field experiments, numerical simulations, and theory.

### 2. DISSIPATION SUBRANGE

Dispersion in the dissipation subrange is closely related to the mixing of scalars and the rate of chemical reaction for reacting scalars in the limit of fast chemistry, the prototypical example being combustion. Moreover, the deformation of microstructures in the flow such as polymer molecules (Jin & Collins 2007, Massah et al. 1993, Terrapon et al. 2004) or drops (Cristini et al. 2003) can be linked to the rate at which points on the surface of the microstructure separate.

We begin with an overview of the theoretical framework in Section 2.1, followed by a review of more recent theoretical advances in Section 2.2. Section 2.3 discusses the relevant DNS and experiments.

#### 2.1. Theoretical Framework

The fluid velocity at two neighboring particle locations can be approximated by a Taylor series expansion to first order, an assumption that is sometimes referred to as the locally linear-flow approximation. The resulting relative velocity is $w = r \cdot \Gamma$, where $\Gamma \equiv \nabla u$ is the velocity gradient tensor in the frame of reference moving with the particle pair. Batchelor (1952b) was the first to recognize that, because the magnitude of the relative velocity is proportional to the separation distance, the dispersion law is exponential in time: $\langle r^2 \rangle \sim r_0^2 \exp(\xi t)$, where $\xi$ is a parameter with units of inverse time. Batchelor & Townsend (1956) argued that $\xi = 2B/\tau_\eta$, where $\tau_\eta \equiv \sqrt{\nu/\langle \epsilon \rangle}$ is the Kolmogorov timescale and $B$ is a dimensionless parameter known as the Batchelor constant. Based on various assumptions regarding the velocity gradient, they estimated $0.35 < B < 0.41$.

Saffman (1960) later analyzed the effect of molecular diffusion on pair dispersion. Under the assumption that $\tau_\eta / \tau_c < 1$ (where $\tau_c$ is the average time between molecular collisions), molecular
diffusion enhances the dispersion rate in an additive manner to leading order. Secondly, molecular diffusion affects the Lagrangian velocity correlation time. Through an analysis of the scalar diffusion equation, Saffman (1960) arrived at the following relationship for relative dispersion with molecular diffusion in the short-time limit (three dimensions):

$$\frac{1}{2} \frac{d(r^2)}{dt} = \frac{B(r^2)}{\tau_\eta} + 6\kappa + 2\kappa \frac{(t - t_0)^2}{\tau_\eta^2},$$

(4)

where $\kappa$ is the molecular diffusivity.

2.2. Modern Theoretical Advances

It is possible to derive the Batchelor constant for the special case of a $\delta$-correlated-in-time velocity. The equation for $(r^2(t))$ is found by substituting the locally linear flow approximation into Equation 2:

$$\frac{1}{2} \frac{d(r^2)}{dt} = \int_0^\infty \langle \Gamma_{ij}(t) \Gamma_{kl}(t') r_j(t) r_k(t') \rangle dt'.$n

(5)

We adopt index notation for clarity, where the Einstein summation rule applies to repeated indices. By assuming that $\Gamma_{ij}(t)$ is $\delta$-correlated in time, Brunk et al. (1997) argued that the motion of the particles during the time the flow is correlated could be neglected, reducing Equation 5 to

$$\frac{1}{2} \frac{d(r^2)}{dt} = (r^2(t)) \int_0^\infty \langle S_{ij}(t) S_{ij}(t') \rangle dt'.$n

(6)

where $S_{ij} = (\Gamma_{ij} + \Gamma_{ji})/2$ is the rate-of-strain tensor. The Lagrangian correlation for $S_{ij}(t)$, under the assumption of a short correlation time, takes the form

$$\langle S_{ij}(t) S_{kl}(t + \tau) \rangle = \frac{\langle e \rangle}{20\nu} \left( \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl} \right) \exp \left( -\frac{\tau}{\tau_S} \right),$$

(7)

where $\tau_S$ is the correlation time for the rate-of-strain tensor. Substituting this into the right-hand side of Equation 6 yields $(e\tau_S/(15\nu))$ or equivalently $B = \tau_S/(15\tau_\eta)$. Lundgren (1981) predicted $\tau_S = \tau_\eta/3$, in agreement with DNS (Yeung & Pope 1988) and experiments (Guala et al. 2007), implying $B = 0.15$. Lundgren (1981) also predicted that the probability density function (PDF) for pair separations is log-normally distributed, a result confirmed by DNS (Girimaji & Pope 1990).

As $\tau_S/\tau_\eta \sim 1$, Brunk et al.’s (1997) assumption of a white-noise process is questionable. Indeed, their subsequent work has shown the effect of finite correlation times on the interparticle collision rate (Brunk et al. 1998a,b). This led Chun et al. (2005) to introduce a new model for the rate-of-strain tensor that relaxes this assumption. They prescribe $S_{ij}$ as a sequence of randomly oriented uniaxial extensional flows of finite (random) duration. The resulting flux of particles at a distance $r$ is given by an integral relationship, which, when equated to the gradient flux approximation, yields $B = 0.093$.

2.3. Simulations and Experiments

Girimaji & Pope (1990) studied the growth rate of infinitesimal lines in isotropic turbulence using Lagrangian velocity gradient data taken from Yeung & Pope’s (1989) DNS. They analyzed $d(\ln r)/dt$ and found that it approached an asymptote corresponding to $B = 0.13$ over the range $38 \leq R_s \leq 93$. They noted the discrepancy with Batchelor & Townsend’s (1956) original estimate and attributed it to the lack of persistence of the rate-of-strain tensor and the tendency of vorticity
to rotate the material line away from the principal extensional axis. A subsequent DNS study confirmed this hypothesis (Huang 1996). Kida & Goto (2002) introduced a different statistical weighting to their analysis and arrived at a slightly higher value of the Batchelor constant, \( B = 0.17 \).

Guala et al. (2005) were the first to investigate pair dispersion in the dissipation subrange experimentally. Using stereoscopic particle tracking (PT), they were able to measure the velocity gradient tensor in the Lagrangian frame of reference. Computing the source term \( \overline{\partial q / \partial t} \), they estimated \( B = 0.097 \). The result is very close to the value found by Chun et al. (2005), whose theoretical estimate was confirmed by a novel simulation of a cloud of so-called satellite particles surrounding each inertial particle that satisfied

\[
\frac{dr_i}{dt} = -A / \tau_q r_i + \Gamma_j r_j, \tag{8}
\]

where \( A \) is an adjustable drift parameter. They showed that the resulting PDF of particle pair separations takes the form \( P(r) \sim (\eta / r)^{3/2} \). By fitting the exponent (for known \( A \)), it was possible to backcalculate \( B \). In the limit \( A \to 0 \), the DNS confirmed their prediction (\( B = 0.093 \)).

3. INERTIAL SUBRANGE

In this section we present the relevant theory pertaining to relative dispersion in the inertial subrange (Section 3.1), followed by a summary of the same for two-dimensional turbulence (Section 3.2). In Section 3.3, finite-scale statistics are discussed as an alternative measure of relative dispersion. We then review recent laboratory experiments (Section 3.4), DNSs (Section 3.5), and experiments in the environment (Section 3.6).

3.1. Theoretical Framework (Three Dimensions)

As the particle pair separates beyond the dissipation scales, the range of motions (or eddy sizes) that move them apart varies with the separation distance, in which eddies of scale \( l \sim r(t) \) are most effective in the process of dispersion (Corrsin 1962). Richardson (1926) initially put forth this notion and suggested the following diffusion equation for relative dispersion in isotropic turbulence:

\[
\frac{\partial q(s,t)}{\partial t} = \frac{1}{d s^{d-1}} \frac{\partial}{\partial s} \left[ s^{d-1} K(s) \frac{\partial}{\partial s} q(s,t) \right], \tag{9}
\]

where \( d \) is the Euclidean space dimension, \( s \) is the sample space coordinate corresponding to the random separation distance between pairs \( r, q(s,t) \) is the PDF of the separation distance between particle pairs subject to the normalization \( \int_0^\infty q(s,t) 4 \pi s^2 ds = 1 \), and \( K(s) [L^{2} T^{-1}] \) is a scale-dependent dispersion coefficient. We note that \( \langle r^3(t) \rangle = \int_0^\infty s^3 q(s,t) 4 \pi s^2 ds \).

Compiling measurements of the effective eddy diffusion coefficient \( K_{\text{eff}} = \frac{1}{n} \sum \frac{d r^2(t)}{d t} \) over a wide range of scales, Richardson found that \( K_{\text{eff}} \sim \langle r^2(t) \rangle^{2/3} \). This result led him to model \( K(s) = \kappa_{R} s^{4/3} \), where \( \kappa_{R} \) has units \([L^{2/3} T^{-1}] \). This since has been referred to as Richardson’s 4/3 law.

Richardson’s 4/3 law also can be viewed as the outcome of what is now widely referred to as K41 theory (Kolmogorov 1941). K41 states that, provided the Reynolds number is sufficiently large and boundary effects are negligible, the energy dissipation rate is determined by the large scales and is independent of the viscosity \( \nu \). The inertial subrange is argued to depend only on the average dissipation rate \( \langle \epsilon \rangle \) (second similarity hypothesis), and the dissipation subrange is uniquely determined by \( \langle \epsilon \rangle \) and \( \nu \) (first similarity hypothesis).

Obukhov (1941), a pupil of Kolmogorov, recognized that the only dimensionally consistent form for a dispersion coefficient that depends only on \( s \) in the inertial subrange is \( K(s) = k_{0} \langle \epsilon \rangle^{1/3} s^{4/3} \), where \( k_{0} \) is a dimensionless constant. With this expression for \( K(s) \), Equation 9
admits a self-similar solution for dispersion from a point source \([i.e., q(s, 0) = \delta(s)]\) with boundary conditions \(q(0, t) = q(\infty, t) = 0\). The result for the three-dimensional case is

\[
q_3(s, t) = \frac{429}{70} \sqrt{\frac{143}{2}} \left[ \frac{1}{\pi (\bar{r}^2(t))} \right]^{3/2} \exp \left[ - \left( \frac{1287s^2}{8(\bar{r}^2(t))} \right)^{1/3} \right],
\]

which implies a mean square displacement,

\[
\langle r^2(t) \rangle = g(\langle \epsilon \rangle) t^3,
\]

where \(g = \frac{1144}{27} k_0^3\). Following convention, we refer to this as the Richardson-Obukhov (R-O) law.

Batchelor (1952a) advocated the use of an explicit time-dependent diffusion coefficient \(K(t) = k_0(\langle \epsilon \rangle) t^2\). If we substitute this expression into Equation 9 and solve it under the same initial and boundary conditions used by Richardson, we arrive at the Batchelor PDF:

\[
q_B(s, t) = \left[ \frac{3}{2\pi (\bar{r}^2(t))} \right]^{3/2} \exp \left[ - \frac{3s^2}{2(\bar{r}^2(t))} \right],
\]

which also implies Equation 11, with \(g = 2k_0\).

Klafter et al. (1987) noted that K41 theory does not predict a unique diffusion coefficient but that any scaling of the form \(K(t) = k_0(\langle \epsilon \rangle) a^b s^c\) satisfying the constraints \(2b + 3c = 4\) and \(a = 1 - c/2\) is both consistent with K41 and yields \(\langle r^2(t) \rangle = g(\langle \epsilon \rangle) t^3\), with \(g = \Gamma(2b/2c) \Gamma(3c/2a)^{1/4}\), where \(\Gamma()\) is the Gamma function. The Richardson PDF is obtained by setting \(b = 0\) (giving \(a = 1/3\) and \(c = 4/3\)), and the Batchelor PDF is obtained by setting \(c = 0\) (giving \(a = 1\) and \(b = 2\)). Intermediate combinations of parameters are also admissible by theory (Hentschel & Procaccia 1984, Okubo 1962); however, the resulting distributions, \(q(s, t)\), for each combination of parameters \((a, b, c)\) are quite different. Other forms for \(q(s, t)\) have been obtained based on closure approximations (Kraichnan 1966).
Boffetta & Sokolov (2002a) remarked that implicit in the idea that relative dispersion is governed by a diffusion equation is the physical assumption that the velocity field is short correlated in time. In fact, for velocity fields that are $\delta$-correlated in time (Kraichnan ensemble), Equation 9 is exact (Falkovich et al. 2001, Kraichnan 1968). In this context, Sokolov (1999) commented that for such a velocity field, Equation 9 with time-dependent and mixed forms of $K(s, t)$ is not admissible, but he also emphasized that it still may apply to other scenarios.

Navier-Stokes turbulence is scale dependent and time correlated (i.e., persistent). Hence it is not clear if a diffusion-type equation governs the process of relative dispersion. Nonlocal effects are important, and it is more likely that the dispersion process is described by an integro-differential equation (Batchelor & Townsend 1956). Such an avenue has been explored with Lévy flights (Klafter et al. 1987; Shlesinger et al. 1987, 1993; Sokolov & Reigada 1999; Sokolov et al. 1999). Lévy walks are a generalization of the homonymous random flights, characterized by instantaneous jumps. In Lévy walks, a probability density is associated with taking a step of length $r$ and duration $t$, thus avoiding the infinite mean square displacements of Lévy flights because after a finite time $t$, a finite number of steps have been taken (Shlesinger et al. 1987, 1993).

Other researchers have introduced higher-order time derivatives to account for finite-time correlations. Monin & Yaglom (1975) considered a differential equation of the type

$$\frac{d}{dt} \langle r^2(t) \rangle = C \Delta q(s, t),$$

where $C$ is a constant and $\Delta$ is the Laplacian. This also gives the $t^3$ result.

Ogasawara & Toh (2006) proposed a differential equation for $q(s, t)$ based on a telegraph process. The essential features of this model are the consideration of finite-time correlations through the introduction of a second-order time derivative term to the left-hand side of Equation 9 and the allowance of a difference in persistence between expansion and compression of the separation vector by adding a drift term multiplied by a flow-dependent coefficient to the right-hand side that, the authors argued, characterizes the average effect of coherent structures in the flow on dispersion. Kanatani et al. (K. Kanatani, T. Ogasawara & S. Toh, submitted manuscript) further explore this model. In a similar fashion, T. Faber & J.C. Vassilicos (submitted manuscript) have allowed for convergence events within the theory of straining stagnation points given by Goto & Vassilicos (2004).

Falkovich et al. (2001) questioned the possibility of establishing general properties of $q(s, t)$. They arrived at this conclusion by rewriting Equation 2 as

$$\frac{1}{2} \frac{d}{dt} \langle |r(t) - r_0|^{2} \rangle = \tau_{w}^{v}(t) \langle |w|^{2} \rangle,$$

(15)

where

$$\tau_{w}^{v}(t) = \int_{0}^{t} \langle w(r') \cdot w(t) \rangle dt'$$

(16)

is the Lagrangian correlation time of the relative velocities. If the separation distances are in the inertial subrange, then $K41$ would imply $\tau_{w}^{v}(t) \sim t$. Under this circumstance, one cannot invoke the central limit theorem and cannot regard the dispersion process as a sum of uncorrelated increments.

Although virtually all theories predict $\langle r^2(t) \rangle \sim t^3$ at intermediate times, clearly there is no consensus on the form of $q(s, t)$ once that regime is reached. We consider this question in the discussions that follow (see Sections 3.4–3.6).

### 3.2. Theoretical Framework (Two Dimensions)

In statistically stationary forced homogeneous and isotropic two-dimensional turbulence, two ranges of relative dispersion are identified:

$$\frac{1}{2} \frac{d}{dt} \langle r^2(t) \rangle \sim \begin{cases} \langle r^2(t) \rangle & \text{for } r(t) \ll l_1 \\ \langle r^2(t) \rangle^{2/3} & \text{for } l_1 \ll r(t) \ll l_0. \end{cases}$$

(17)
where \( l_i \) is the energy-injection length scale, \( l_0 \) is the largest length scale in the flow, and \( l_n \equiv \tilde{\eta}^{-1/3} \nu_{\text{diss}}^{1/2} \) is the enstrophy-dissipation length scale, where \( \tilde{\eta} = v((\nabla \omega)^2) \) is the dissipation rate of enstrophy \( \frac{1}{2} \omega^2 \), where \( \omega \) is vorticity. The first result of Equation 17 is associated with the direct enstrophy cascade and is attributed to Lin (1972), whereas the second result is for the inertial subrange of the inverse energy cascade, analogous to Richardson’s \( 4/3 \) law in three-dimensional turbulence. Upon integration, we have

\[
\langle r^2(t) \rangle = \begin{cases} 
\langle r^{(0)} \rangle \exp (c' \delta) & \text{for } r(t) \ll l_i \\
\langle r^{(0)} \rangle + g'(\epsilon) \epsilon^3 & \text{for } l_i \ll r(t) \ll l_0,
\end{cases}
\] (18)

where \( c' \) and \( g' \) are nondimensional constants, and \( \epsilon \) is a timescale defined by the enstrophy dissipation rate and enstrophy in the local and nonlocal cases, respectively (Babiano et al. 1990).

3.3. Finite-Scale Statistics

A complication in analyzing the time evolution of mean particle pair separations is the contamination from particle pairs that lie outside the regime of interest. For example, particle pairs in the dissipation subrange that happen to separate quickly may enter the inertial subrange, thereby modifying the apparent scaling for the dissipation subrange. An alternative approach to studying particle dispersion is to consider particle pair statistics at fixed particle separations or intervals instead of at fixed times. With so-called finite-scale statistics, it is possible to ensure particle statistics are gathered exclusively from the subrange of interest. Below we describe two approaches to obtaining dispersion rates using finite-scale statistics.

3.3.1. Finite-size Lyapunov exponent. Artale et al. (1997) introduced the finite-size Lyapunov exponent (FSLE) as an alternative measure of turbulent relative dispersion by extending concepts from dynamical systems theory. The maximum Lyapunov exponent \( \lambda_{\text{max}} \) is a measure of the average rate of exponential separation \( \delta(t) = \delta(0) \exp(\lambda_{\text{max}} t) \) of infinitesimally close trajectories in a chaotic dynamical system (Aurell et al. 1997). The multiscale nature of turbulence then permits one to define the FSLE by considering the average response of trajectories as a function of the separation distance. Finite thresholds \( \delta^{(n)} \) are defined such that \( \delta^{(n)} = \rho^n \delta^{(0)} \), where \( n = 1 \ldots n_{\text{max}}, \delta^{(0)} = \rho_0 \) is the initial threshold level, \( n_{\text{max}} \) defines the maximum threshold, and \( \rho = \delta^{(n+1)}/\delta^{(n)} > 1 \) is the threshold rate. The time required for a given pair to move from \( \delta^{(0)} \) to \( \delta^{(n+1)} \) is denoted by \( T_{\rho}(\delta^{(n)}) \).

The mean exit time (or doubling time) is given by \( (T_{\rho}(\delta^{(0)})) \). The FSLE is then defined as

\[
\lambda_L(\delta) = \frac{\ln \rho}{T_{\rho}(\delta^{(0)})},
\] (19)

which in the limit of vanishing \( \delta \) gives the maximum Lyapunov exponent

\[
\lambda_{\text{max}} = \lim_{\delta \to 0} \lambda_L(\delta),
\] (20)

The FSLE has the advantage over finite-time statistics such as \( \langle |r(t) - r_0|^2 \rangle \) in that the latter has contributions from pairs that are in different regimes of the dispersion process for any given time. This is especially true at low to moderate \( R_e \), at which there is no clear separation of scales. The mean exit time, conversely, has contributions from pairs that predominantly lie between \( \delta^{(0)} \) and \( \delta^{(n+1)} \), effectively removing contamination due to crossover from other scales (Boffetta & Sokolov 2002a). The threshold rate should be chosen judiciously so as to allow for separation of the different regimes. We note that \( \langle r^2(t) \rangle \sim t^{\gamma} \) corresponds to \( \lambda(\delta) \sim \delta^{-1/\gamma} \).
In the diffusion limit of a delta-correlated velocity field, it is possible to relate the mean exit time to the Richardson constant from the first-passage problem of the Richardson diffusion equation (Boffetta & Sokolov 2002b). In three dimensions (Boffetta & Sokolov 2002a),

$$g = \frac{143}{81} \left( \frac{\rho^2}{3} - 1 \right)^{\frac{1}{3}} \frac{\delta^2}{\langle \epsilon \rangle \langle \langle T_\rho \rangle \rangle^{\frac{1}{3}}}.$$  \(\text{(21)}\)

### 3.3.2. Scale-dependent diffusivity.

T. Faber & J.C. Vassilicos (submitted manuscript) have proposed the use of the mean square diffusivity (see Equation 2) conditioned on the separation scale as a contamination-free measure of relative dispersion. We generalize this approach by conditioning on

$$\left\langle \frac{1}{2} \frac{d}{dt} |r(t) - r_0|^2 \right\rangle = \langle (r(t) - r_0) \cdot w(t) \rangle,$$  \(\text{(22)}\)

which in the inertial subrange takes the form

$$\langle (r(t) - r_0) \cdot w(t) \rangle = \begin{cases} \left( \frac{11}{2} C_2 \right)^{1/2} \left( \langle \epsilon \rangle r_0 \right)^{1/3} \Delta & \text{for } t \ll t_B \\ \frac{3}{2} g \langle \epsilon \rangle^{1/3} \Delta^{3/3} & \text{for } t_B \ll t \ll T_L \end{cases}$$  \(\text{(23)}\)

for the Batchelor and R-O regimes, respectively. The results above follow from Equations 12 and 13. The advantage of this approach over the FSLE is that \(g\) and \(C_2\) can be determined directly. If further conditioning is made on the sign of \([r(t) - r_0] \cdot w(t)\), then one can distinguish between \(g_f\) and \(g_b\), where the indices \(f\) and \(b\) refer to forward and backward dispersion, respectively.

### 3.4. Laboratory Experiments

The laboratory offers a controlled environment wherein experiments can be repeated many times under nearly identical conditions. The single greatest disadvantage of laboratory measurements is the limited range of Reynolds numbers that can be achieved. Recent breakthroughs in three-dimensional stereoscopic particle imaging and high-speed detection have enabled the simultaneous measurement of multiple particle trajectories in a turbulent flow field, a technique known as PT (Malik et al. 1993, Virant & Dracos 1997). Below we review modern PT experiments performed in laboratory turbulent flows.

#### 3.4.1. Three-dimensional turbulence.

Virant & Dracos (1997) were the first to present results on the relative dispersion of particle pairs in a laboratory setting. They used essentially the same tracking system described by Maas et al. (1993) in an open channel flow. By tracking a cloud of quasi-neutrally buoyant particles \((d \ll \eta)\) released in the flow centerline, they found that \(g(s, t)\) in the homogeneous direction, perpendicular to the mean flow, was in good agreement with \(q_B\); \(R_\lambda = 101\) was relatively low for inertial subrange scaling to hold; the initial cloud size was larger than the integral scale; and there was strong shear (Ott & Mann 2000). Thus the relative particle pair separation PDF is expected to be Gaussian, as the particles are essentially wandering independently, and therefore this does not support a Batchelor PDF for relative dispersion in the inertial subrange. The measurement volume was approximately \(600\eta(120L_l)\) in the homogeneous direction, where \(L_l\) is a measure of the Lagrangian integral length scale.

Ott & Mann (2000) measured three-dimensional relative dispersion in turbulence generated by a pair of oscillating grids in a water tank \((R_\lambda \approx 100)\). For particles initially separated by \(\sim 10\eta\), \((r^2(t))\) initially exhibited non-R-O-like scaling, which the authors attributed to a \(t^3\) dependence,
consistent with the Batchelor regime. However, Franzese & Cassiani (2007) have argued that a $t^3$ scaling may arise spuriously from the representation of a nonzero initial separation on a log-log scale. For the appropriate identification of the Batchelor regime, the initial separation must be subtracted [i.e., $\langle |r(t) - r_0|^2 \rangle$ should be plotted preferably in coordinates compensated for Batchelor scaling]. To enhance a possible R-O scaling, the authors performed a time shift $\tau = t - T_0$ in their data, where $T_0$ is given by the root (zero crossing) of a linear fit of $\langle |r(t)|^2 \rangle^{1/3}$ in the inertial subrange. Rewriting the R-O law as $\langle r^2(\tau) \rangle = g(c)\tau^3$, a $\tau^3$ scaling range is observed for roughly $10\tau_\nu$. The measured Richardson constant for this range is $g = 0.5$. Particles were followed for up to roughly $10\tau_B$, and the measurement volume was a cube of length $7L$ ($600\nu$). Computed pair-separation PDFs within the inertial subrange scaling agree well with $q_R$. The procedure of time shifting the data by the zero crossing $T_0$ has been criticized by Ouellette (2006), who showed that for a given power law $\sim t^{\eta}$, the procedure can yield a range $\sim t^\gamma$ for any $\gamma$.

Bourgoin et al. (2006) and Ouellette et al. (2006) were able to track hundreds of particles at $R_s$ in excess of 800, with very high temporal resolution (up to 27,000 Hz). Turbulence was generated by coaxial counter-rotating baffled disks in a cylindrical water tank. They achieved the high frame rate using state-of-the-art CMOS (complementary metal oxide semiconductor) cameras. Their results show unequivocal evidence of the Batchelor regime over two decades of $t/\tau_B$ at $R_s \approx 815$, as seen in Figure 1a, in which the data are compensated to yield a plateau at 1 in the Batchelor range. The excellent collapse is lost if $\langle |r| \rangle^{2} - \langle |r_0| \rangle^{2}$ is plotted, which suggests $\langle \mathbf{w} \cdot \mathbf{r} \rangle \neq 0$. This is an important observation and should be considered in other experiments. The curves deviate from the Batchelor regime at approximately $t/\tau_B = 0.07$. The decrease in the slope beyond $t/\tau_B = 0.07$ suggests a transition to a regime $t^\gamma$ with $\gamma < 2$, in disagreement with a crossover to the R-O regime. This is attributed to the influence of normal diffusion that occurs when particle pairs separate beyond the integral scale. A bias caused by the finite measurement volume also is not ruled out. The pair-separation PDFs $q(|\Delta \mathbf{s}| = |s - s_0|, t)$ for several initial separations are shown in Figure 1b–g. For the smallest separation distance, there is better agreement with $q_R$ and with increasing initial separation distance, the distribution approaches $q_B$. The agreement with $q_B$ may be a symptom of contamination from the large scales, as the initial separation shown in Figure 1c ($387\nu$ to $430\nu$) is in the middle of the inertial subrange ($60 \lesssim r/\eta \lesssim 1000$), as identified by K41 scaling of the second-order velocity structure function. The ratio of the measurement volume length to the integral length scale is $L_{\text{col}}/L \approx 0.7$, where $L = u'/\langle \epsilon \rangle$ and $u'$ is the fluctuating root-mean-square velocity. [The value of $L = 7.1 \text{ cm}$ was taken from an earlier paper characterizing this facility (Voth et al. 2002).]

Berg et al. (2006) measured relative dispersion in a water tank with turbulence generated by eight rotating propellers located in the corners ($R_s = 172$). By using a temporal shift similar to that of Ott & Mann (2000), the authors found a scaling region that seems consistent with the R-O law. However, they did not claim to have found true R-O scaling. They also used their data to calculate the pair-separation PDF, finding good agreement with $q_B$ (see Figure 2). The initial particle pair separations are not entirely within the inertial subrange. The measured Richardson constant for forward and backward dispersion is $g_f = 0.55 \pm 0.05$ and $g_b = 1.15 \pm 0.05$, respectively. It is argued that the lack of R-O scaling in the experiments of Bourgoin et al. (2006) and Ouellette et al. (2006) may be related to a bias introduced by their small measurement volume ($L_{\text{col}}/L \approx 0.7$). In the current study $L_{\text{col}}/L_{\text{int}} \approx 2.5$, and only pairs that start within a central subvolume are considered.

3.4.2. Two-dimensional turbulence. Two-dimensional turbulence in the laboratory has been investigated mainly in soap films and in stratified layers of conducting fluid over solid substrates. In the former case, obstacles in the soap film may generate gridlike turbulence, whereas in the
Figure 1

(a) Mean square relative dispersion for 50 different bins of initial separations, ranging from 0–1 mm (≈0–43η) to 49–50 mm (≈2107–2150η), compensated by Batchelor scaling (C₂ = 2.1). (b–g) Probability density functions (PDFs) of pair separations. The red straight line is the Richardson PDF, whereas the blue curved line is the Batchelor PDF. The symbols show the experimental measurements. Each plot shows a different initial separation; for each initial separation, PDFs from 20 times ranging from 0–20τ/η are shown. \( R_λ = 815 \), \( Δr(t) = |r(t) - r₀| \), and 1 mm ≈ 43η. Figure adapted from Ouellette et al. 2006.
Figure 2
(a) Pair-separation probability density functions (PDFs) for forward (gray dots) and backward dispersion (green dots) for pairs with \( r_0 = 12 - 16\eta \). The red straight line is the Richardson PDF, and the blue curved line is the Batchelor PDF. (b) \( \langle r(t)^2 \rangle / \langle r^2(t) \rangle \) versus \( t/t_B \) for forward dispersion. The different sequences correspond to different initial separations: \( r_0 \) increasing upward from \( r_0 = 4\eta \) to \( r_0 = 28\eta \) in bins of \( 4\eta \). The horizontal lines are the Richardson prediction at 0.67 and the Gaussian prediction at 0.85. (c) Same as in panel b, but for the backward case. Figure adapted with permission from J. Berg, B. Lüthi, J. Mann, S. Ott. 2006. Backwards and forwards relative dispersion in turbulent flow: an experimental investigation. Phys. Rev. E 74:016304. Copyright (2006) by the American Physical Society.
latter, electromagnetic forcing is the preferred technique. A recent review is provided by Kellay & Goldburg (2002). Here we describe only experiments that have measured two-particle dispersion. In addition to its likely relevance to intermediate- to large-scale geophysical flows (Charney 1971, McWilliams 1983, Provenzale 1999, Read 2001, Rhines 1979), Boffetta & Celani (2000) claimed that two-dimensional turbulence is an ideal framework to verify the R-O regime because of the absence of intermittency of the Eulerian velocity increments in the inverse energy cascade (Boffetta et al. 2000, Paret & Tabeling 1998), even though M.K. Rivera & R.E. Ecke (submitted manuscript) have found anomalous scaling of the Lagrangian structure function exponents by applying the extended self-similarity methodology (Benzi et al. 1993). The lower dimensionality of the flow allows for easier flow characterization and enables a larger separation of scales.

Jullien et al. (1999) generated two-dimensional turbulence in a cell filled with two thin layers of NaCl solution of different concentrations in a stable configuration. Permanent magnets located in the solid substrate were oriented such that their magnetization axis was in the vertical direction. The stirring was accomplished by Laplace forces originating from the interaction of the magnetic field with random current impulses driven across the cell. They measured the surface velocity field using particle image velocimetry (PIV) at regular intervals (∼10 Hz). The measured flow field was then used to advect numerical particles in time with a fourth-order Runge-Kutta scheme. They observed a two-dimensional inverse energy cascade that can be characterized by a Reynolds number, \( R_{el} \equiv u' / l_1 \sim 5 \), where \( c \) is a damping coefficient [T\(^{-1}\)]. The results (Figure 3) display a convincing \( t^3 \) range, with a Richardson constant \( g \sim 0.5 \). The pair-separation PDF has an exponential distribution of the form \( q(\xi) \sim \exp[-a\xi^\beta] \), where \( \xi = s / \langle r^2(t) \rangle^{1/2} \), \( a \sim 2.6 \), and \( \beta = 0.5 \pm 0.10 \). The Richardson result is \( \beta = 2/3 \). Another important result from this experiment is the measurement of the Lagrangian correlation time of the relative displacements \( \tau_L (t) \sim 0.60t \), showing that for any given time the particles are correlated for more than half of the time since their release.

![Figure 3](image-url)

**Figure 3**

Rivera & Ecke (2005) explored a technique similar to Jullien et al.’s (1999), except that they replaced the bottom fluid layer with Fluorinert FC-77, which is immiscible and almost two times heavier than water, allowing for a higher Reynolds number \( R_{el} \approx 20 \) without vertical mixing between the layers. The velocity field was again obtained via PIV. By advecting numerical particles, they found an approximate \( t^3 \) range, albeit for initial separations and for mean scales that lie below the injection scale (enstrophy range), at which no such \( t^3 \) range is expected. The pair-separation PDFs show a form similar to \( qR \). By using the mean doubling time to measure relative dispersion, they showed in their data a clear exponential regime up to scales of \( r/L_1 \approx 1 \), followed by a power-law range with an exponent of 0.83 instead of the Richardson prediction of 2/3 (corresponding to \( t^{3/4} \) instead of \( t^3 \)).

### 3.5. Direct Numerical Simulation

DNS refers to the solution of the three-dimensional, constant-property Navier-Stokes equations with sufficient numerical resolution to ensure that all turbulent scales are resolved adequately. Not unlike laboratory experiments, DNS can achieve only a limited range of Reynolds numbers. Conversely, it offers the most complete description of the flow available, superior to any experiment.

Yeung (1994) was the first to study relative dispersion with a 128³ DNS of forced homogeneous isotropic turbulence. In spite of the modest Reynolds number \( R_s \approx 90 \), significant conclusions could be drawn. Yeung found relative dispersion consistent with the Batchelor regime. No clear \( t^3 \) range was observed or sought. For particles with \( r_0/\eta \approx 1 \) and for times \( t/\tau_{\eta} \lesssim 30 \), the particle separation and relative velocity were highly intermittent, as measured by the non-Gaussian skewness and flatness factors. Forcing was accomplished using Eswaran & Pope’s (1988) stochastic scheme.

Boffetta & Sokolov (2002a) found a modest \( t^3 \) range for relative particle dispersion in their 256³ DNS at \( R_s \approx 200 \). Energy was held constant in the two lowest wave-number modes, and fluid velocities at particle positions were obtained via linear interpolation. The initial particle pair separation was 1/256 of the box size \( L_{box} = 2\pi \), such that \( r_0/\eta \approx 1.8 \) (G. Boffetta, personal communication). The authors found good agreement with \( q_R \) at short times, which seems to relax to the Gaussian form for increasing times. When dispersion was measured via the mean exit time, they found a broader R-O range than that obtained by the traditional measure of the mean-squared separation distance. Assuming a Richardson form of \( q(t, t) \), they found the Richardson constant to be \( g \approx 0.55 \). A simple model led to Lagrangian intermittency corrections for the doubling time that are related to the scaling exponents of the Eulerian longitudinal velocity structure functions. The authors performed a similar DNS study in the inverse energy cascade of two-dimensional turbulence (Boffetta & Sokolov 2002b), for which the general conclusions above are valid. In this work the authors derived the long-time asymptotic form for the mean exit time PDF by assuming R-O scaling and showed that the DNS PDFs, when normalized by the mean exit time, are self-similar and in good agreement with the theoretical result \( \sim \exp(-c T/\langle T \rangle) \), where \( c \) is a numerical constant.

Ishihara & Kaneda (2002) used a procedure similar to Ott & Mann’s (2000), in which \( (r^2(t))^{1/3} \) is plotted as a function of \( t - t_B \) for initial separations ranging from 2.8\( \eta \) to 22.4\( \eta \), none of which lies in the inertial subrange. Following this procedure, they estimated a Richardson constant \( g \approx 0.7 \). For this 1024³ DNS, \( R_s \approx 283 \), and forcing was accomplished through a variable negative viscosity in the wave-number range \( k < 2.5 \) that fixed the kinetic energy.

Building on earlier work (Yeung 1994, 2001), Yeung & Borgas (2004) found evidence of an R-O regime with \( g \approx 0.83 \) from a 512³ DNS at \( R_s \approx 230 \) and for \( r_0/\eta = 16 \) (this initial separation was the closest available to inertial subrange conditions). Of great importance was the recognition of the temporal variability in the spatially averaged dissipation rate \( \epsilon \) in DNS, which undergoes a low-frequency oscillation in time owing to the forcing. Their recommendation is to use \( \epsilon \) averaged over...
the time of the dispersion process in place of the long-time average when evaluating Lagrangian universal scaling laws. It is also recognized that this temporal variability is strongly influenced by the choice of the forcing scheme (Eswaran & Pope 1988).

Goto & Vassilicos (2004) used two-dimensional DNS to investigate the importance of the density of straining stagnation points \( n_p \), on the process of turbulent relative dispersion, an idea originally presented by Dávila & Vassilicos (2003) who used kinematic simulations. The argument is that pairs tend to separate violently when they encounter straining stagnation points, a view supported by the individual tracks from experiments in the environment (LaCasce & Ohlmann 2003, Ollitrault et al. 2005) and in the laboratory (Jullien et al. 1999, Ott & Mann 2000, Virant & Dracos 1997). From three-dimensional kinematic simulations, DNS, and a wind-tunnel experiment, Dávila & Vassilicos (2003) found a fractal scaling \( n_p \sim (L/\eta)^{D_s} \) for \( L/\eta \gg 1 \), where \( D_s = 2 \) is the fractal dimension of the spatial distribution of straining stagnation points. Goto & Vassilicos (2004) further explored the idea and related the exponent \( \gamma \) of the mean square separation of particles \( \sim r^\gamma \) to the fractal dimension \( D_s \) (an Eulerian quantity): \( \gamma = 2d/D_s \), where \( d \) is the Euclidean space dimension, \( D_s \) satisfies \( p + 2D_s/d = 3 \), \( p \) is the power-law exponent of the turbulent kinetic energy spectrum in the inertial subrange \( [E(k) \sim k^{-5/3}] \), and \( k \) is the wave number. They found that this relation holds for \( p = 5/3 \) and \( D_s = 4/3 \) obtained from DNS of two-dimensional turbulence \( (d = 2) \). They noted that stagnation points are not Galilean invariant and that the application of the above theory should be performed in a reference frame that maximizes the persistence of the streamline topology in a statistical sense. The assumption is made that such a frame of reference exists and that in isotropic turbulence it is the frame of reference for which the mean fluid velocity is zero. A posteriori DNS supports this argument (Goto et al. 2005).

Biferale et al. (2005) used the mean exit time to obtain an estimate for the Richardson constant of \( g = 0.50 \pm 0.05 \) for a 1024\(^3\) DNS at \( R_i \sim 280 \). This method assumes the validity of the Richardson PDF and is in good agreement with the estimate \( g = 0.47 \) obtained by fitting a straight line to \( \langle r^2(t) \rangle^{1/3} \) for an initial separation of \( r_0/\eta = 2.5 \) (well below the inertial subrange). The finite-scale statistics do not show dependence on initial pair separation, which for this study lies below \( r_0/\eta = 20 \). The separation PDF, measured for \( r_0/\eta = 1.2 \) pairs, agrees well with \( \delta_R \) in the range 40–70\( r_0 \). The authors confirm observations of previous studies at similar \( R_i \) (Boffetta & Sokolov 2002a, Ishihara & Kaneda 2002, Yeung & Borgas 2004), such as strong intermittency in the dispersion process for particle pairs initially closely separated. Numerical results confirm that the correlation times \( \tau_{\epsilon}(t) \) and \( \tau_{\epsilon}'(t) \) are proportional to the time \( t \) (see Figure 4). In this study, forcing is accomplished by maintaining the total energy constant in the first two wave numbers, and fluctuations in \( \epsilon \) were found to be at most 10% of the mean value.

Sawford et al. (2008) presented dispersion results for Reynolds numbers up to \( R_i \approx 650 \) (2048\(^3\)), similar to those measured in experiments by Bourgoin et al. (2006) and Ouellette et al. (2006). The results are compared with earlier DNS (Biferale et al. 2005, Ishihara & Kaneda 2002, Yeung et al. 2006). Following Yeung & Borgas’s (2004) recommendation, the dissipation rate is averaged over the duration of the dispersion process only:

\[
\langle \epsilon^{1/3} \rangle = \frac{1}{T} \int_0^T \epsilon^{1/3}(t') dt'.
\] (24)

This approach yields an improved collapse of data when compensated by inertial subrange scaling laws that depend on \( \langle \epsilon^{1/3} \rangle \). The mean square separation distance for the data compiled in

\(^1\)These relations have been explored experimentally in a two-dimensional electromagnetically driven laminar multiscale flow, in which precise control over forced stagnation points is possible. In the experiment \( p \approx 2.5 \), \( D_s \approx 0.5 \), and \( d = 2 \) such that \( p + 2D_s/d = 3 \) (Rossi et al. 2006).
this study, compensated by R-O scaling, is shown in Figure 5a. Except for \( R_0 = 38 \) and \( r_0/\eta \geq 16 \), the data for different \( R_0 \) at the same \( r_0/\eta \) collapse well for small times, during which a ballistic regime is expected for all \( r_0/\eta \) considered. Furthermore, if the time is compensated by \( t_B \) (see Figure 5b) for \( r_0/\eta > 16 \) and \( R_0 \geq 240 \), the curves collapse at all times, irrespective of \( r_0 \). For \( r_0/\eta \leq 1 \) and all \( R_0 \), a minimum in the compensated coordinates occurs before the R-O regime. Sawford (2001) and Boffetta & Sokolov (2002a) attributed this minimum to the contamination of the inertial subrange by dissipation subrange effects. For \( R_0 \approx 650 \), the inertial subrange extends from \( r/\eta \approx 16 \) to \( r/\eta \approx 300 \). Based on the arguments presented in Section 3.1, R-O scaling is expected to hold after a time \( t \gg t_B \), and because the initial separation is at the lower end of the inertial subrange, the largest \( t^3 \) range for the data should occur for \( r_0/\eta = 16 \). This is confirmed in Figure 5a. However, contrary to expectation, \( t^3 \) scaling occurs for \( t/T_\lambda > 1 \), which suggests that the requirement given in Equation 13 for \( t \) is too stringent and that the determining factor is the separation scale, not the timescale. We note that \( t_B = (r_0/\eta)^2/\nu \), such that, even for the cases \( 64 \leq r_0/\eta \leq 256 \) (which lie in the inertial subrange), R-O scaling is expected to manifest only after \( t \gg 16T_\lambda \) and \( t \gg 40\tau_\eta \), respectively. Figure 5b convincingly shows R-O scaling, as demonstrated by the collapse of the curves for different initial separations within the inertial subrange for \( t/t_B > 1 \). This data set constitutes the best evidence available of an R-O regime. The method of the \( \langle t^3 \rangle \) plot described by Ott & Mann (2000) and also used by Ishihara & Kaneda (2002) is not recommended because of the ambiguity in the choice of the appropriate range of times over which to apply the fit. Sawford et al. (2008) argued that a more accurate estimate can be obtained by plotting \( \frac{\langle \hat{r}^2 \tau^2 \rangle^{1/3}}{\nu} \). For \( r_0 \) in the inertial subrange, a plateau independent of \( r_0 \) gives the Richardson constant. Figure 5c shows a compilation from several DNS data sets of estimates of \( g \) as a function of \( R_0 \) from these local slope plots along with a power law and exponential fit, which give \( g = 0.55 \) and \( g = 0.57 \), respectively, as \( R_0 \rightarrow \infty \).

### 3.6. Experiments in the Environment

Environmental flows produce the highest Reynolds numbers on Earth; hence theories such as the R-O law, which was derived in the limit \( R_0 \rightarrow \infty \), have the greatest chance of being tested.

![Figure 4](image-url)
However, there are complications in natural flows, such as inhomogeneities of the turbulence, stratification and other buoyancy effects, Coriolis forces, two-dimensional effects due to boundary conditions, and the vastness of the range of scales of the measurement (millimeters to kilometers) that challenge experimentalists to find measurable conditions in which the theory may apply. Despite these challenges, historically it has been in the environment that most dispersion experiments have been performed.
Richardson (1926, 1929) reported the first measurements of particle pair dispersion, in the form of $K_{\text{eff}}$. Common traits of these early experiments include both the limitation of the experimental methods employed and the ingenuity of the experimenters. Among the fluid tracers used in Richardson’s first experiments were dandelion seeds, smoke puffs from cigarettes, and balloons of various sizes. In a celebrated experiment, Richardson & Stommel (1948) measured $K_{\text{eff}}$ by throwing carefully cut pieces of parsnips off Blairmore Pier in Loch Long, Scotland, and measuring their relative positions with an optical device. Based only on two values of $\langle r^2 \rangle^{1/2}$, the authors reported a power law $K_{\text{eff}} \sim (\langle r^2 \rangle^{1/2})^{1.4}$.

Many experiments were conducted in the following years in the atmosphere and oceans, most of which are described in a comprehensive chapter in Monin & Yaglom (1975, pp. 556–67). To a large extent, the experiments of this period do not offer unequivocal evidence of any regime, although they undoubtedly support the scaling $\langle r^2(t) \rangle \sim t^\gamma$, with $\gamma \in [2, 3]$.

Below we review what we believe to be the most relevant experiments of relative dispersion in the environment. We purposely omit experiments that are not strictly Lagrangian, based on the premise that such experiments must rely on indirect measures of two-particle dispersion.

### 3.6.1. Atmosphere.

The EOLE experiment was conducted from October 1971 through January 1972. Approximately 480 constant-volume 200-mbar-level balloons were tracked in the mid-latitudes of the southern hemisphere by the French satellite Eole (Morel & Bandeen 1973). Among the experiment’s objectives was the measurement of particle pair dispersion, through which a possible form of the energy spectrum in the 100–1000-km range could be inferred. Morel & Larcheveque (1974) found agreement with a $k^{-3}$ spectrum based on measurements of the observed exponential growth of $\frac{1}{2} \int_0^t \frac{dr}{t^\gamma}$, with an $e$-folding time of 2.7 days. This exponential growth was followed by a linear dependence in $t$. The EOLE data have been reanalyzed by Lacorata et al. (2004), who instead used the FSLE to investigate the scale dependence of relative dispersion. The $\lambda_L$ values display a substantially shorter exponential regime ($e$-folding time of 0.4 days) than found originally, and contrary to a linear dependence, the latter interpretation resulted in an R-O regime.

Er-El & Peskin (1981) analyzed the TWERL (Tropical Wind, Energy Conversion, and Reference Level) experiment, conducted between June 1975 and December 1976 (Julian et al. 1977). In the TWERL experiment, 383 tetroons at the 150 mbar level were released in tropical and mid-latitudes of the southern hemisphere and subsequently tracked via the Nimbus 6 satellite. An exponential range of relative dispersion was found for the mid-latitude releases. Although the authors claimed to have found a $t^\gamma$ dependence following the exponential range, the scatter in the data is large, and we believe this claim to be too strong.

### 3.6.2. Ocean.

LaCasce & Ohlmann (2003) used surface-drifter tracks from SCULP (Surface Current and Lagrangian Drift Program) (Ohlmann & Niiler 2005) to study relative dispersion regimes in the Gulf of Mexico. The drifters were tracked by Doppler ranging of the ARGOS satellite system five to seven times a day. Exponential growth was found in a range of scales from 5 km to 50 km (1 to 10 days) with an $e$-folding time of approximately 2 days. At larger times, a power-law scaling was observed, $t^\gamma$, where $\gamma = 2.2$ if fixed time statistics were used $\langle r^2(t) \rangle$, but $\gamma = 3$ when the FSLE was adopted. The internal Rossby radius $R_{\text{int}}$ is approximately 45 km. Because drifters are constrained to the ocean surface, the resulting flow is not divergence-free (Falkovich et al. 2001), and therefore predictions for two-dimensional turbulence do not strictly apply. This, however, does not rule out the possibility of such a scaling (Schumacher & Eckhardt 2002). Within the range of scales available, a linear regime was not found.
Ollitrault et al. (2005) analyzed a small subset of quasi-Lagrangian tracks from the TOPOGULF experiment (Arhan et al. 1989, de Verdière et al. 1989). During TOPOGULF, 26 subsurface floats at the 700-dbar level were deployed in the mid-latitude range of the North Atlantic, and their positions were tracked between 1983 and 1989. The subset of float data revealed that for initial separations less than $R_{\text{int}} = 25$ km, there is evidence of an exponential regime ($t < 6$ days), followed by a $t^3$ regime from scales of 40 km to 300 km (20 to 60 days) after which a linear $t$ regime was observed (see Figure 6). This is consistent with a two-dimensional direct enstrophy cascade at scales smaller than $R_{\text{int}}$ and an inverse energy cascade at scales larger than $R_{\text{int}}$, although the authors note that there are other possible explanations for these trends, and because the plots were constructed under the assumption of an R-O regime (by performing the appropriate time shift), they cannot prove its existence. Particle-separation PDFs were also computed. For initially close pairs, the PDFs are non-Gaussian, whereas for initially distant pairs, there is stronger resemblance to a Gaussian PDF, in agreement with an earlier surface-drifter experiment off the northern California coast (Davis 1985a,b).

4. DISCUSSION

Despite the insightful beginning provided by the classic papers, there remain serious questions about the dispersion of particle pairs in turbulence. This is, in no small part, due to the many challenges faced by experimentalists and numerical simulators in making measurements that can accurately and reproducibly test the theory. Early experiments in the environment have shown the strongest evidence of the R-O regime; however, there are a number of important caveats. The data often have significant scatter, which may be attributable to spatial inhomogeneities in the turbulence parameters and/or measurement errors. Vertical and horizontal mean shear (Bowden 1965; Csanady 1969; Kullenberg 1972; Randerson 1972; Saffman 1962a,b; Young et al. 1982) that may enhance dispersion is often not measured. This effect may even dominate turbulent dispersion under the conditions $R_i \sim 1$ and $ST \gg 1$, where $R_i$ is the Richardson number, $S$ is
the mean shear rate, and $T$ is the integral timescale. Moreover, drifters used as surface markers in oceans or lakes are not capable of following the inherently three-dimensional turbulent motion. Their motion on the surface is not equivalent to a two-dimensional, incompressible turbulence owing to convergence or divergence events that render the surface flow compressible (Falkovich et al. 2001, Okubo 1970). Recent laboratory experiments and numerical simulations have shown that dispersion on a free surface exhibits a reduced scaling exponent in comparison with the classical R-O value (Cressman & Goldburg 2003, Cressman et al. 2004). Finally, in studies that report an R-O regime, the values of the Richardson constant can vary by orders of magnitude (Monin & Yaglom 1975).

Nature’s laboratory, while providing an enormous range of scales, exhibits an interplay of phenomena that must be well understood and accounted for before these results can be used to test theory. Batchelor (1959), reflecting on this very issue at the 1958 International Symposium on Atmospheric Diffusion and Air Pollution at Oxford University, wrote, “needed is a more refined version of the Richardson-Stommel parsnip experiment, carried out with careful regard for the conditions under which the similarity theory may be expected to apply, and performed if possible in the laboratory rather than in the atmosphere.”

Advances in PIV and PT have made this a reality. Overall, the turbulence in a laboratory apparatus is reproducible and better controlled. DNS in two or three dimensions also offers a well-controlled environment in which to study two-particle dispersion.

Results from experimental measurements appear to be converging on the properties of $\langle r^2(t) \rangle$ in the inertial subrange. For example, Jullien et al.’s (1999) two-dimensional experiments that used PIV to measure the velocity field, and numerically solved for the motion of particles in the flow, yielded convincing R-O scaling. However, the same level of support was not found in Rivera & Ecke’s (2005) more recent experiment of the same nature, in which they found a power-law range with a reduced exponent ($\sim t^{1.4}$). One major difference between these experiments is that Jullien et al. performed forcing on the bottom layer, whereas Rivera & Ecke performed forcing on the top layer, on which the particles were tracked. It seems that further investigation of the influence of the forcing is warranted. The clever approach of numerically integrating the particles circumvents some of the problems found with conventional PT (e.g., initialization of particle pairs at controlled separations). Some three-dimensional experiments find $t^3$ scaling (Berg et al. 2006, Ott & Mann 2000), but at Reynolds numbers that are too low to justify attributing this to the R-O regime. Bodenschatz and coworkers (Bourgoin et al. 2006, Ouellette et al. 2006, Xu et al. 2008) have reached the highest Reynolds numbers in three-dimensional laboratory experiments to date. Although they have found an extensive Batchelor range, they have yet to observe the R-O regime. This can be attributed to a bias caused by the measurement volume or to the Reynolds number, which, although the largest so far, still may not be large enough.

DNS studies of pair dispersion suffer from many of the same limitations as laboratory experiments (Boffetta & Sokolov 2002a, Yeung 1994, Yeung & Borgas 2004). Biferale et al. (2005) showed how the use of finite-scale statistics has promise for mitigating some of these problems. Most convincing is Sawford et al.’s (2008) DNS that, at 2048$^3$, had a sufficient inertial subrange to allow the particle pairs to be initialized well within the inertial range. It is encouraging that all these studies seem to be converging on a Richardson constant in the range $0.5 \leq g \leq 0.6$, much improved relative to measurements in the environment (Monin & Yaglom 1975).

However, the state of affairs for the pair-separation PDF is far less clear. Richardson’s use of a diffusion equation can be rigorously justified only for a velocity field that is $\delta$-correlated in time and if the process is self-similar at all scales, which does not hold in turbulent flows. A further critique, raised by Sawford et al. (2005), is that $\delta$-correlated velocity fields (or a diffusion equation for relative dispersion) cannot differentiate between forward and backward dispersion.

Sokolov
presented heuristic arguments that convincingly show the importance of time correlations in the analysis of relative turbulent dispersion. If one uses a unique kinematic parameter to specify the flow over some range of scales, as is the case in K41 theory, then the mean-square separation distance exhibits the same scaling for a \( \delta \)-correlated velocity field, for the ballistic regime, and for two-dimensional isotropic turbulence. Differences among these flows manifest in the form of the PDF of pair separations. The question of whether a self-similar form for the PDF even exists has been raised (Falkovich et al. 2001). However, in spite of the apparent flaws, the majority of recent experiments and simulations indicate a PDF most similar to the Richardson form, but our understanding of this remains incomplete.

5. CONCLUDING REMARKS

The classical framework created by Richardson (1926), placed in the context of K41 by Obukhov (1941), and extended and clarified by Batchelor (1950) has held up remarkably well over time. Nothing has effectively challenged the existence of the R-O regime. Similarly, there has not been an experiment that has unequivocally confirmed R-O scaling over a broad-enough range of time and with sufficient accuracy.

Looking to the future, it appears that tools to solve these puzzles are emerging. We therefore think it is appropriate to reflect on the lessons learned over the past eight decades of research. Up to now, the primary focus of most experiments has been the search for the R-O regime in the form of \( t^3 \) scaling. Because nearly all self-consistent theories for the inertial subrange yield a \( t^3 \) power law (Sokolov 1999), we believe that future experiments and simulations should place greater emphasis on Richardson’s constant and the PDF of particle pairs. The latter in particular appears to be a better discriminator among the various theories. A second challenge is to ensure particle pairs are well within the inertial range, avoiding contamination from the dissipation subrange that can lead to a spurious \( t^3 \) regime. To that end, we note that finite-scale statistics offer a clear advantage over analyzing \( \langle r^2(t) \rangle \). We offer a few additional observations and recommendations specific to each of the four approaches to studying pair dispersion.

5.1. Environmental Measurements

Environmental measurements have the complication that the turbulence may not be homogeneous. Consequently, the mean flow and turbulence should be well characterized over the domain of the experiment and for the duration of the measurement. A second challenge is the vastness of the domain to be studied. Experimentalists can exploit developments in remote sensing based on global positioning systems and so-called smart dust that contains embedded electronics to transmit its particle location (Butler 2006).

5.2. Laboratory Measurements

The goal of laboratory measurements should be to push the Reynolds number to the largest value possible. Supercooled liquid helium, owing to its low molecular viscosity, can achieve much higher Reynolds numbers than conventional fluids (Niemela et al. 2000). Likewise, a new wind tunnel under construction in Göttingen (at the Max Planck Institute for Dynamics and Self-Organization, directed by Professor Eberhard Bodenschatz) will use compressed SF\(_6\) as its working fluid to achieve \( R\_\text{\lambda} \) of the order of 10,000. These facilities (and possibly others) can provide a well-conditioned environment to study R-O scaling. The accuracy in the measurement of the Richardson constant may be affected by errors in the measurement of the dissipation rate \( \epsilon \). It
is recommended that more than one approach to measuring $\epsilon$ be used to provide an objective estimate of any potential bias errors. The finite measurement volume may cause a significant bias in Lagrangian statistics because longer tracks will be associated with slower particles. This has been observed in Mordant et al.’s (2004) single-particle measurements. Assessment of the bias can be observed by conditioning statistics on track length. Figure 7 illustrates this effect using DNS data from Biferale et al. (2005). Notice that statistics conditioned on relatively small track lengths show no evidence of an inertial subrange $t^3$ scaling, whereas as the conditioning length increases, the more rapidly separating pairs can be observed for a longer time, revealing what appears to be R-O scaling. Concerns over Lagrangian stationarity also have been raised (Ott & Mann 2005). Further investigation of these effects is recommended.

5.3. Direct Numerical Simulations

As with laboratory measurements, the issue of pushing the Reynolds number higher is paramount. There has been steady growth in this parameter over the past three decades, and future peta-scale computers (Bement 2007), or even lattice-Boltzmann-based grid computations (Celani 2007), offer the hope of extending DNS beyond the present-day maximum of $4096^3$ (Kaneda et al. 2003), corresponding to $R_\infty \sim 1000$. DNS with $8192^3$ and beyond will be possible on peta-scale computers with $10^5$–$10^6$ processors. Although DNS may lag behind the highest values of $R_\infty$ in experiments, the fact that the turbulence in most DNS is forced apparently causes the turbulence to reach the asymptotic limit at a lower Reynolds number than is required in an experiment with decaying turbulence (Antonia & Burattini 2006).

As relative dispersion inevitably involves scales that approach the energy-containing eddies, the role of forcing must be quantitatively assessed. There are deterministic (Witkowska et al. 1997) and stochastic (Eswaran & Pope 1988) forcing algorithms that can be compared to determine what effect (if any) they have on two-particle dispersion. A related issue is the low-frequency fluctuation in the dissipation rate that is found in forced turbulence. We recommend the approach used by Sawford et al. (2008), who replaced the globally averaged dissipation rate with its value averaged over the time of each dispersion calculation.

Figure 7

(a) Mean square separation distance versus $t/\tau_\eta$ conditioned on volume size for pairs with $r_0 = 2.4\eta$ at $R_\lambda = 284$. Cubic volume of length: $0.59L$ (orange), $0.78L$ (green), $1L$ (light blue), $1.5L$ (dark blue), $2L$ (red), and $2L$ with periodicity (purple). (b) Same as in panel a, but compensated by $(t/\tau_\eta)^3$ to reveal the Richardson regime. Data from a $1024^3$ direct numerical simulation (Biferale et al. 2005).
Homogeneity in DNS is approximated by periodic boundary conditions. Periodicity also allows for the analysis of particle pairs with separations that exceed the box size; however, it generates the unphysical situation that two particles separated by integer multiples of the box size in all three directions will have the same velocity. Yeung (1994) argued that this does not have a significant effect on Lagrangian statistics because these occurrences are rare and the turbulence is continuously evolving, such that the influence on two-time statistics is even weaker. This argument is supported by the results in Figure 7, which compare the mean square separations computed with and without periodicity. The former provide a more convincing R-O regime, most likely due to improved statistics at large separations. Nevertheless, it seems reasonable that this issue should be analyzed more carefully in future DNS.

5.4. Theory

The persistence of turbulence challenges the use of a diffusion equation. In the dissipation subrange, persistence gives rise to nonlocal diffusion. Approaches based on a telegraph scheme show promise (Chun et al. 2005). In the inertial subrange, the issue may be more serious. In particular, the objection raised by Falkovich et al. (2001) concerning the existence of a self-similar PDF should be investigated further. They argued that the correlation time for the relative velocity between two particles grows indefinitely. Franzese & Cassiani (2007) predicted a similar result. Nevertheless, experimental measurements of PDFs show good agreement with the Richardson or Batchelor predictions (Ouellette et al. 2006), but this does not necessarily confirm R-O scaling. For example, the Gaussian prediction of Batchelor equally would apply to normal diffusion, and therefore may represent contamination from the large scales. This needs to be further analyzed from a theoretical perspective.

Analyses to date have focused on the statistics of particle pairs (e.g., mean separation rate, PDF), with comparatively little attention paid to the underlying dynamics. One exception is Goto & Vassilicos’s (2004) model that connects dispersion with the local strain near stagnation points. The predictions are consistent with R-O scaling, but the proposed mechanism goes further to connect an Eulerian property (density of straining stagnation points) to the Lagrangian dispersion process. Further analysis of the dynamics of particle dispersion along these lines is warranted. Statistical tests should then be devised to evaluate the validity and importance of various proposed mechanisms.

The role of intermittency (Frisch 1995) on pair dispersion remains unclear. Intermittency corrections have been proposed that lead to a scaling of the form $\langle r^2(t) \rangle \sim t^k (t/T_E)^\kappa$, with $\kappa \geq 0$ and $t/T_E \ll 1$ (Hentschel & Procaccia 1983, 1984; Shlesinger et al. 1987). The magnitude of the correction depends on the model used to account for intermittency. However, the procedure of incorporating such corrections contains a certain level of ambiguity. Crisanti et al. (1987) have advocated a correction for $\langle r^2(t) \rangle$, where $p = 2$, with no anomalous scaling for higher exponents, whereas others have proposed corrections that are absent for $p = 2$, but exist for $p > 2$ (Boffetta et al. 1999, Novikov 1989). It seems that verification of such corrections will require even larger $R_\lambda$.

Moreover, the role of particle inertia on two-particle dispersion should be explored. In particular, the regime of small Stokes numbers can be analyzed by methods currently used with fluid particles (with minimal further adaptation).

The most important application of two-particle dispersion is in understanding the motion of passive scalars in the atmosphere and oceans. Although the Reynolds numbers of natural flows are large, conditions rarely match the assumptions made in the theory (e.g., isotropic turbulence, absence of two-dimensional effects). On a fundamental level, as discussed by Warhaft (2008), the question remains as to whether experiments should be designed to mimic the complexity of natural mechanisms.
phenomena or if they should strip away the more complex reality to isolate a particular aspect of the whole. The former approach is referred to as holistic, whereas the latter is referred to as reductionist. Conversely, what is the utility of a theory that can be realized only in laboratory flows in which all complications have been carefully removed? Ultimately, the theory must be generalized to account for the nonidealities found in nature. That is why observations in nature, although not as controlled as in the laboratory, remain the ultimate demonstration of the utility of the theory.

**DISCLOSURE STATEMENT**

The authors are not aware of any biases that might be perceived as affecting the objectivity of this review.

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